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Categorification and Interpretation: Beyond the Formal

Mathematics is often described as a "formal" science as opposed to sciences studying certain specific empirical "content" like physics, chemistry, biology or any of their sub-divisions. The idea behind this description is that mathematical objects are forms (or more precisely "pure" or "idealized" forms) of things found in Nature. In this talk I analyze the relevant notion of form and claim the following: Categorification significantly widens the traditional notion of mathematics by extending it "beyond the formal".

The traditional notion of mathematical "form" is that of an invariant under certain reversible transformations (an invariant of a group). So this notion depends on the corresponding class of transformations. For example, in a metrical context the form of "circle" is defined up to motions and scale transformations, while in a purely topological context "circle" is defined up to homeomorphism. (This, of course, makes a difference: in a metrical context circular and oval forms are different but topologically they are the same.) This geometrical pattern applies to algebra thanks to reversibility of the substitution: so one may call formula x+y=y+x a "common algebraic form" of numerical expressions 1+2=2+1, 2+3=3+2, etc. . D. Hilbert in his "Grundlagen der Geometrie" applies this notion of form to a geometrical theory as a whole through his "axiomatic method": the idea is to describe a given theory only up to isomorphism leaving it to the user to choose his or her favorite model. Although this method, generally speaking, doesn't work - I mean the presence of so called "non-standard" models - Hilbert's formalist approach (in the wide sense just mentioned) still remains very influential in mathematics. Categorification changes this fundamental pattern and the associated intuition of "form" by the simple fact that it assumes the notion of non-reversible transformations (morphism) as primitive and defines reversible morphisms as a special case. Thus in the context of functorial semantics first proposed in Lawvere's thesis the usual worry of ruling out non-standard models makes no sense, and the very distinction between a theory and its model changes its usual meaning. Algebraic and geometric properties in a categorical context reveal profound links unnoticed before. In my talk I shall analyze these and some other consequences of "dropping the reversibility" via categorification.