# Foundations of Axiomatic Mathematics

## Lecture 1. Axiomatic Architecture of Euclid's *Elements*

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1. Axiomatic Architecture of Euclid's *Elements*
2. Two Axiomatic Styles : Hilbert and Gentzen
3. Axiomatic Method in Topos theory and in Homotopy Type theory

# Plan of Lecture 1

# Ian Müller 1974 on Euclid

I know of no logic which accounts for [. . . ] inference in its Euclidean formulation. One 'postulates' that a certain action is permissible and 'infers' the doing of it, he., does it.

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- Definitions :
  play the same role as *axioms* in the modern sense ; ex. *radii of a circle are equal*
- Common Notions (Axioms) :
  play the role similar to that of logical rules restricted to mathematics : cf. the use of the term by Aristotle
- Postulates :
  non-logical constructive rules

# Common Notions 1-3

A1. Things equal to the same thing are also equal to one another.

A2. And if equal things are added to equal things then the wholes are equal.

A3. And if equal things are subtracted from equal things then the remainders are equal.

# Common Notions (continued)

Euclid's Common Notions hold *both* for numbers and magnitudes (hence the title of "common"); they form the basis of a regional "mathematical logic" applicable throughout the mathematics but not beyond mathematics.

# Euclid's Ax.1 as a Rule

$$\frac{A = C, \; B = C}{A = B}$$

# Euclid's Ax.1 as aProposition : Carroll's Regress

$$(A = C\&(B = C) \rightarrow (A = B)$$

$$\frac{(A = C)\&(B = C)\ ,\ (A = C)\&(B = C) \rightarrow (A = B)}{A = B} \quad (\textit{modes ponens})$$

# Postulates 1-3 :

P1 : to draw a straight-line from any point to any point.

P2 : to produce a finite straight-line continuously in a straight-line.

P3 : to draw a circle with any center and radius.

# Postulates 1-3 (continued) :

Postulates 1-3 are NOT propositions even in their grammatical form They are not first truths. They are extra-logical rules.

# Operational interpretation of Postulates

| Postulates | input | output |
|---|---|---|
| P1 | two points | straight segment |
| P2 | straight segment | straight segment |
| P3 | straight segment and its endpoint | circle |

# Propositional interpretation of Postulates

P1m (modal) : Given two (different) points it is always *possible* to produce a straight segment from one given point to the other given point.

P1e (existential) : Given two (different) points there *exists* a straight segment having these given points as its endpoint.

The propositional interpretation of Postulates is not compatible with the deductive structure of the *Elements*!

# Shared Structure (Proclus) :

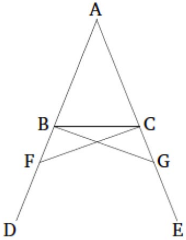
- *enunciation* :
- *exposition*
- *specification*
- *construction*
- *proof*
- *conclusion*

# Theorem 1.5 :

[*enunciation :*]
For isosceles triangles, the angles at the base are equal to one another, and if the equal straight lines are produced then the angles under the base will be equal to one another.

# Theorem 1.5 (continued) :

[*exposition*] :
Let $ABC$ be an isosceles triangle having the side $AB$ equal to the side $AC$ ; and let the straight lines $BD$ and $CE$ have been produced further in a straight line with $AB$ and $AC$ (respectively). [Post. 2].



# Theorem 1.5 (continued) :

[*specification* :]
I say that the angle $ABC$ is equal to $ACB$, and (angle) $CBD$ to $BCE$.

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[*specification* :]
I say that the angle *ABC* is equal to *ACB*, and (angle) *CBD* to *BCE*.

[*construction* :]
For let a point *F* be taken somewhere on *BD*, and let *AG* have been cut off from the greater *AE*, equal to the lesser *AF* [Prop. 1.3]. Also, let the straight lines *FC*, *GB* have been joined. [Post. 1]

# Theorem 1.5 (continued) :

[*proof* :]
In fact, since *AF* is equal to *AG*, and *AB* to *AC*, the two (straight lines) *FA*, *AC* are equal to the two (straight lines) *GA*, *AB*, respectively. They also encompass a common angle *FAG*. Thus, the base *FC* is equal to the base *GB*, and the triangle *AFC* will be equal to the triangle *AGB*, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) *ACF* to *ABG*, and *AFC* to *AGB*. And since the whole of *AF* is equal to the whole of *AG*, within which *AB* is equal to *AC*, the remainder *BF* is thus equal to the remainder *CG* [Ax.3]. But *FC* was also shown (to be) equal to *GB*. So the two (straight lines) *BF*, *FC* are equal to the two (straight lines) *CG*, *GB* respectively, and the angle *BFC* (is) equal to the angle *CGB*, while the base *BC* is common to them

# Theorem 1.5 (continued) :

[*proof* (continued) :]
Thus the triangle *BFC* will be equal to the triangle *CGB*, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus *FBC* is equal to *GCB*, and *BCF* to *CBG*. Therefore, since the whole angle *ABG* was shown (to be) equal to the whole angle *ACF*, within which *CBG* is equal to *BCF*, the remainder *ABC* is thus equal to the remainder *ACB* [Ax. 3]. And they are at the base of triangle *ABC*. And *FBC* was also shown (to be) equal to *GCB*. And they are under the base.

# Theorem 1.5 (continued) :

[*conclusion* :]
Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

Mind the difference between monstration and demonstration (deixis kai apodeixis). *Quod erat demonstrandum* is an erroneous Latin translation of Euclid's expression. The correct one is *quod erat monstrandum*.

# Problem 1.1 :

[*enunciation :*]
To construct an equilateral triangle on a given finite straight-line.

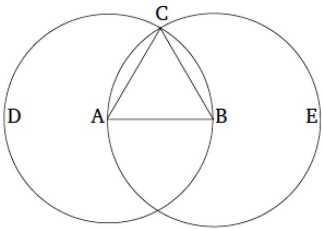
# Problem 1.1 (continued) :

[*exposition* :] Let AB be the given finite straight-line.

[*specification* :] So it is required to construct an equilateral triangle on the straight-line $AB$.

# Problem 1.1 (continued) :

[*construction* :]
Let the circle *BCD* with center *A* and radius *AB* have been drawn [Post. 3], and again let the circle *ACE* with center *B* and radius *BA* have been drawn [Post. 3]. And let the straight-lines *CA* and *CB* have been joined from the point *C*, where the circles cut one another, to the points *A* and *B* [Post. 1].



# Problem 1.1 (continued) :

[*proof* :]
And since the point $A$ is the center of the circle $CDB$, $AC$ is equal to $AB$ [Def. 1.15]. Again, since the point $B$ is the center of the circle $CAE$, $BC$ is equal to $BA$ [Def. 1.15]. But $CA$ was also shown (to be) equal to $AB$. Thus, $CA$ and $CB$ are each equal to $AB$. But things equal to the same thing are also equal to one another [Axiom 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) $CA$, $AB$, and $BC$ are equal to one another.

# Problem 1.1 (continued) :

[*conclusion* :]
Thus, the triangle $ABC$ is equilateral, and has been constructed on the given finite straight-line $AB$. (Which is) the very thing it was required <u>to do</u>.

Problems and Theorems are intertwined and form a joint deductive structure rather than two parallel structures. A typical theorem involves a non-trivial construction that uses earlier construction ; a typical problem involves a proof that is based on earlier proven theorems. Compare Curry-Howard correspondence.

Term "proposition" as a common name for Problems and Theorems is missing in Euclid's *Elements* and is not recorded in Proclus' *Commentary* and other reliable sources.

# Proclus on Problems and Theorems

Some of the ancients, however, such as the followers of Speusippus and Amphinomus, insisted on calling all propositions "theorems", considering "theorems" to be a more appropriate designation than "problems" for the objects of the theoretical sciences, especially since these sciences deal with eternal things [..]. Thus it is better, according to them, to say that all these objects exist and that we look on our construction of them not as making, but as understanding them [..]

# Proclus on Problems and Theorems (continued)

Others, on the contrary, such as the mathematicians of the school of Menaechmus, thought it correct to say that all inquiries are problems but that problems are twofold in character : sometimes their aim is to provide something sought for, and at other times to see, with respect to determinate object, what or of what sort it is, or what quality it has, or what relations it bears to something else." (Comm.Eucl., p. 63-64).

Puzzle : Aristotle uses word "axiom" for logical principles (non-contradiction, excluded middle, perfect syllogism) and occasionally he refers suing the same term to Euclid's Ax.1-3.

Solution : Aristotle's idea is to introduce new principles of reasoning, which are "common" like mathematical axioms (which are common for arithmetic and geometry) but also cover material substances (and hence can be used in any reasoning rather than only mathematical reasoning.

# Universal Mathematics

There are certain mathematical theorems that are universal, extending beyond these substances [viz., numbers and magnitudes]. Here then we shall have another intermediate substance separate both from the Ideas and from the intermediates [i.e., mathematical objects], a substance which is neither number nor points nor spatial magnitude nor time. (Met. 1077a)

# Mathematical and Logical Axioms (1) :

We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called axioms and into [the general issue of] essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For [logical] axioms hold of everything that there is but not of some particular genus apart from others. (Met. 1005a)

# Mathematical and Logical Axioms (2) :

Since the mathematician too uses commons [i.e., axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however qua being. (Met. 1061b)

# Inadequacy of Aristotle's Syllogistic to Geometrical Proofs

Let *A* be two right angles, *B* triangle, *C* isosceles. Then *A* is an attribute of *C* because of *B*, but it is not an attribute of *B* because of any other middle term ; for a triangle has [its angles equal to] two right angles by itself, so that there will be no middle term between A and B, though AB is matter for demonstration." (An. Pr. 48a33-37)

# Kolmogorov & Fomin 1976)

[*enunciation :*]
Any closed subset of a compact space is compact
[*exposition :*]
Let $F$ be a closed subset of compact space $T$
[*specification :*
I say that $F$ is a compact space]

# Kolmogorov & Fomin 1976)(continued) :

[*construction* :]
[Let] $\{F_\alpha\}$ [be] an arbitrary centered system of closed subsets of subspace $F \subset T$.
[*proof* :]
[E]very $F_\alpha$ is also closed in $T$, and hence $\{F_\alpha\}$ is a centered system of closed sets in $T$.
Therefore $\cap F_\alpha \neq \emptyset$. By Theorem 1 it follows that $F$ is compact.
[*conclusion* :]
Thus any closed subset of a compact space is compact. (Which is) the very thing it was required to show.

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- Problems and Theorems share the same formal structure and form a joint deductive order ;
- Euclid's pattern of mathematical reasoning has been amazingly stable throughout centuries and it remains at work in today's mathematical practice ;
- but it is not adequately represented in standard formalizations of mathematical reasoning.

# Teaser

HoTT and Univalent Foundations of mathematics represent in a wholly new formal setting important aspects of Euclid's pattern.

Thank you !