Foundations of Axiomatic Mathematics

Lecture 2: Two Axiomatic Styles: Hilbert and Gentzen

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Hilbert 1894

Among the appearances or facts of experience manifest to us in the observation of nature, there is a peculiar type, namely, those facts concerning the outer shape of things, Geometry deals with these facts [..]. Geometry is a science whose essentials are developed to such a degree, that all its facts can already be *logically deduced* from earlier ones.

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- Any two distinct points of a straight line completely determine that line
- If different points A, B belong to straight line a and to straight line b then a is identical to b.

The distinction between logical and non-logical concepts plays a fundamental role in Hilbert's 1899 axiomatics because it provides a sense of being formal for his axiomatic theories. The form in point is a logical form. That means that logical semantics (which is not explicitly construed in this framework!) is rigidly fixed and the non-logical semantics is left variable. A formal mathematical theory is grounded in logic and logic alone. Logic in this context is thought of as a system of rules for handling propositions.

Hintikka 1997 on Hilbert-style axiomatic method

The basic clarified form of mathematical theorizing is a purely logical axiom system.

What are possible "fillings" for non-logical elements of a formal theory?

- usual intuitions
- interpretations of the given formal theory in other informal theories (ex.: arithmetical models of geometric theories or one geometric theory in another one)
- "thought-things" (*Gedankendinge*)

All of these contribute to the standard notion of *model* due to Tarski. One may wonder how a combination of these three very different concepts can be coherent.

Hilbert 1918: Axiomatizing Logic

[I]t appears necessary to axiomatize logic itself and to prove that number theory and set theory are only parts of logic. This method was prepared long ago (not least by Frege'sprofound investigations); it has been most successfully explained by the acute mathematician and logician Russell. One could regard the completion of this magnificent Russellian enterprise of the axiomatization of logic as the crowning achievement of the work of axiomatization as a whole.

Notice a conceptual gap (Hintikka). Whatever the axiomatization of logic may be it can not be an axiomatization in 1899 sense!

Hilbert 1927: Intuition Strikes Back!

No more than any other science can mathematics be founded by logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought.

Hilbert 1927: Intuition Strikes Back!

If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication.

Hilbert 1927: Intuition Strikes Back!

And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

Hilbert 1927: Intuition Strikes Back!

[I]n my theory contentual inference is replaced by manipulation of signs [ausseres Handeln] according to rules; in this way the axiomatic method attains that reliability and perfection that it can and must reach if it is to become *the basic instrument of all research* (italics mine - A.R.).

A "logical" proof comes back to a symbolic (albeit not genuinely geometrical) construction . . .

but the intended logical semantics of these symbolic construction plays a crucial role anyway. If one drops this semantics then, in particular, Hilbert's idea of formal consistency proof makes no sense. Why the possibility or impossibility to construct a symbolic expression of form $p\&\neg p$ is so important? Only because the intended logical semantics makes it important.

Logical consequence according to Tarski

Propositional form B is a logical consequence of propositional forms $A_1, \dots, A_n$ iff every interpretation I of the given language, which makes $A_1, \dots, A_n$ into true propositions $A_1^I, \dots, A_n^I$ makes B into true proposition B^I, in symbols $A_1, \dots, A_n \models B$.

Tarski's conception of consequence explains *rules* of inference like $A_1, \ldots, A_n \models B$ via a meta-theoretical *propositions*. The resulting "Platonic" conception of logic does not qualify the notion of rule as fundamental.

The notion of rule from this point of view is syntactic: syntactic deductions of form $A_1, \ldots, A_n \vdash B$ symbolically reflect *facts* of the matter that certain relations of logical consequence hold.

Rules versus Axioms

"The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. "(Gentzen 1935)

Natural Deduction and Sequent Calculus

Gentzen 1935: semantics via introduction rules

The introductions [i.e. introduction rules] represent, as it were, the 'definitions' of the symbol concerned.

Implicit Definitions

Compare Gentzen's idea to use syntactic rules as a form of implicit definitions with Hilbert's use of axioms as implicit definitions. The two approaches may appear to be very similar but in fact they are not.

Think of usual axioms of Group Theory. These axioms serve as a definition in the following sense: any structure, which satisfy the axioms, i.e., serves as their model, is a group. Model theory, which originates from Tarski's pioneering works, explains away the *satisfaction relation* in terms of truth-conditions.

Implicit Definitions (continued)

What kind of entity X can possibly "satisfy" a *rule* or a system of rules, so one could claim that the rules "define" X in some reasonable sense? How the satisfaction relation (if it can be used here at all) has to be construed in this case?

General Proof Theory

In model theory, one concentrates on questions like what sentences are logically valid and what sentences follow logically from other sentences. But one disregards questions concerning how we *know* that a sentence is logically valid or follows logically from another sentence. General proof theory would thus be an attempt to supplement model theory by studying also the evidence or the process - i.e., in other words, the proofs - by which we come to know logical validities and logical consequences. (Prawitz 1974)

Meaning Explanation

is analogous to program compiler: translation of the given syntax into elementary self-evident steps of reasoning (Martin-Löf 1985)

Proof-theoretic Semantics

PTS is not denotational. It does not assign certain entities to certain symbols. It assigns to symbols (and first of all to logical symbols, i.e., to logical constants) their *meaning*, which is not construed in this case as an entity. The procedure of such an assignment is called after Martin-Löf 1985 the *meaning explanation* and consists, roughly, of the explication of computational content of logical constructions in terms of their building blocks, which are presented in a self-explanatory canonical form.

Terminological Issues

Is it justified to call the Gentzen-style formal approach *axiomatic* even if it is rule-based rather than axiom-based?

I tend to answer in positive having in mind Arsitotle's use of the term "axiom". Axiomatic method is more general than we learn it from Hilbert.

Are Hilbert- and Gentzen-style axiomatic presentations inter-translatable?

Only limitedly and only at the syntactic level. Since an axiom is a rule with the empty set of premises the Gentzen-style is more basic. Under certain strong conditions a sequent calculus is deductively equivalent to a Hilbert-style theory with the only rule of modus ponens (Krupski).

However there is no regular procedure known to me that support a *semantic* translation between the two axiomatic styles. In particular, it is unclear how extra-logical rules in theories can be systematically dispensed with.

Open Question

What is behind the usual rendering of Euclid's constructive Postulates into first-order sentences? MLTT suggests that such a translation is an overkill (since quantifiers are represented with Π- and Σ-types).

Gentzen-style beyond Logic?

Until recently the Gentzen-style rule-based formal approach has been used only in purely logical calculi (with the only exception of formal arithmetic). Unlike Hilbert Gentzen never attempted to use his version of axiomatic method in all areas of mathematics or in sciences. Univalent Foundations is the first attempt to use this approach more widely in mathematics. I suggest that it can be also effectively used in Knowledge Representation in general.

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- Allow for managing proofs of very different kinds, which include both mathematical arguments and empirical evidences;
- Support the representation of all sorts of *methods* including extra-logical mathematical methods, experimental methods, etc.;
- Are computer-friendly, i.e., allow for encoding into suitable program languages.

Genetic method: Hilbert 1900

The idea: Mathematical objects are built from other such objects. More complex objects are built from simpler ones.

Hilbert's example

Dedekind Cuts and Cochy sequences. Both are "built from" natural numbers.

Notice that neither of the two "constructions" is constructive in any the usual senses of the word (Turing, Bishop, Markov, et. al)!

Genetic method: Hilbert 1900

Despite the high pedagogic and heuristic value of the genetic method, for the final presentation and the complete logical grounding of our knowledge the axiomatic method deserves the first rank.

Hilbert & Bernays 1934

The term axiomatic will be used partly in a broader and partly in a narrower sense. We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics. [..].

Hilbert & Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* [in the narrow sense?] and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory only as a *genus* of things, an axiomatic theory refers to a fixed system of things [..] given as a whole. Except for the trivial cases where the theory deals only with a finite and fixed set of things, this is an idealizing assumption that properly augments the assumptions formulated in the axioms.

Hilbert & Bernays 1934

Euclid does not presuppose that points or lines constitute any fixed domain of individuals. Therefore, he does not state any existence axioms either, but only construction postulates. (op. cit. p. 20a)

Objectual Constructivity

A constructive axiomatic theory, generally, comprises extra-logical rules similar to Euclid's Postulates, which allow for building of and operating with non-propositional objects. While usual notions of constructivity in logic and mathematics specify such rules in one way or another I leave it wholly open *what* such rules may or should be.

The theory of *Elements*, Book 1 qualifies as constructive in *that* sense. As we shall see HoTT does so too.

Ryle 1945 on Knowing How to Reason Logically

[T]he intelligent reasoner is knowing rules of inference whenever he reasons intelligently'. Yes, of course he is, but knowing such a rule is not a case of knowing an extra fact or truth ; it is knowing how to move from acknowledging some facts to acknowledging others. Knowing a rule of inference is not possessing a bit of extra information but being able to perform an intelligent operation. *Knowing a rule is knowing how.* (The emphasis is added by the authors.)

Propositional language

Definition

Propositional language is a calculus with a distinguished finite set of symbols called *connectives*, which includes connective "$\rightarrow$"; other symbols are called *propositional variables*.

Propositional theory

Definition

Propositional theory is a set T of formulae closed under application of the standard *modus ponens* (MP).

Hilbertian Theories

Definition

A propositional axiomatic theory is called *Hilbertian* when it comprises as theorems all formulae of the form $K_{A,B}$ and $S_{A,B,C}$ where

$$K_{A,B} \doteq \quad A \rightarrow (B \rightarrow A)$$
$$S_{A,B,C} \doteq \quad (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

and has exactly one rule, namely MP.

Deduction Property

Definition

Theory T is said to have the *Deduction Property* (DP for short) if $\Gamma, F \vdash G$ entails $\Gamma \vdash F \rightarrow G$ for all Γ, F and G.

Theorem

An axiomatic propositional theory is Hilbertian if and only if it has the Deduction Property.

Carroll's Regress

$A \vdash B$ if and only if $\vdash A \rightarrow B$
$A, A \rightarrow B \vdash B$ if and only if $A \vdash (A \rightarrow B) \rightarrow B$
$A, (A \rightarrow B) \rightarrow B \vdash B$ if and only if $A \vdash ((A \rightarrow B) \rightarrow B) \rightarrow B$
....

The Deduction Property does not help one to avoid using logical *rules* in reasoning. However one my argue that Tarski semantical conception of logical consequence reduces one's *knowledge* of logical rule $A \vdash B$ to knowing a (model-theoretic) *proposition* $M_{A \models B}$.

Objection:

Generally, $M_{A \vdash B}$ can *not* be known independently of $A \vdash B$. Tarski's model-theoretic semantics takes into account truth-values of propositions but *not* how these truth-values are known or can be known.

Constructive Hilbertian Sequent Calculus

Let T be a Hilbertian theory. We associate now with T a typed sequent calculus CT as follows:

- Types of CT are all formulae of T;
- With each axiom A of T associate constant c^A, which we interpret as the trivial derivation of A in T. In the cases of axioms $(K_{A,B})$ and $(S_{A,B})$ we use the established notation and denote the corresponding constants as $k^{A \to (B \to A)}$ and $s^{(A \to (B \to C)) \to ((A \to B) \to (A \to C))}$ omitting the upper index when this cannot cause a confusion.

Constructive Hilbertian Sequent Calculus (continued)

- Terms of CT correspond to derivations in T; these terms are built from variables and constants with a single binary operation (multiplication), which is an application of rule MP. Each such term determines a unique binary tree such that its internal nodes are marked by MP and its leaves correspond either to T-derivations of axioms or to variables. Rules of CT specify when this tree is the correct tree of derivation from hypotheses in T.

Constructive Hilbertian Sequent Calculus (continued)

- Sequences of CT are expressions of form

$$x_1 : F_1, \ldots, x_n : F_n \vdash t : F,$$

where $x_1, \ldots x_n$ are mutually different variables, $F_1, \ldots F_n, F$ are types (formulae) and t is a term. Sequences determine the same trees but comprise an additional markup: they put label F to the root and attach mark F_i to each leave x_i , which signifies that x_i is a variable over derivations of formula F_i). The obtained tree can get new isolated nodes marked by variables, which are not elements of term t; leaves, which are not in the list $x_1, \ldots x_n$ may remain unmarked.

Axioms and rules of CT

- $x_1{:}\,F_1, \ldots, x_n{:}\,F_n \vdash c^A{:}\,A$, where A is an axiom of T,
- $x_1{:}\,F_1, \ldots, x_n{:}\,F_n \vdash x_i : F_i$,
- $$\frac{x_1{:}\,F_1, \ldots, x_n{:}\,F_n \vdash u{:}\,(F \to G) \quad x_1{:}\,F_1, \ldots, x_n{:}\,F_n \vdash v{:}\,F}{x_1{:}\,F_1, \ldots, x_n{:}\,F_n \vdash (u \cdot v){:}\,G}.$$

Lemma

Every derivable sequence $x_1\colon F_1, \ldots, x_n\colon F_n \vdash t\colon F$ *in* CT *corresponds to a unique derivation* $F_1, \ldots F_n \vdash F$ *in* $\mathbf{T}$*. Each derivation* $F_1, \ldots F_n \vdash F$ *in* $\mathbf{T}$ *corresponds to a unique term* t *such that its associated sequence* $x_1\colon F_1, \ldots, x_n\colon F_n \vdash t\colon F$ *is derivable in* CT*.*

Constructive Deduction Theorem

Theorem

If sequence $x_1\colon F_1, \dots, x_n\colon F_n, x\colon F \vdash t : G$ is derivable in CT, then there exists term u such that sequence $x_1\colon F_1, \dots, x_n\colon F_n \vdash u\colon (F \to G)$ is also derivable.

An Open Problem:

Is there a systematic procedure that translates a theory with extra-logical rules into a theory in which all rules are logical (with some reasonable criterion of logicality? Cf. Hilbert's axiomatic reconstruction of Euclid's geometry.

THANK YOU