Voevodsky's view on the (In)Consistency of Peano Arithmetic

A tribute to the memory of V.V.(1966-2017)

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Volodya

Timeline (the last 15 years):

- 2002: Fields Medal for the proof of Milnor Conjecture in Motive theory;
- 2003-2005: Unsuccessful attempts to work in Mathematical Biology and other applied areas;
- 2006: A very short note on Homotopy λ-calculus: the first appearance of the idea of HoTT;
  Foundations of Mathematics and Homotopy theory: IAS faculty (inaugural) lecture;
- 2010: Univalent Foundations (multiple talks);
  What if the current foundations of mathematics are inconsistent? Lecture at the celebration of the 80s anniversary of the IAS. September, 25
- 2010-2017: Work on the UF with a focus on the Initiality Conjecture. Last talk at the Logic Colloquium 2017 in Stockholm.

GII

Theorem (Gödel):

It is impossible to prove the consistency of any formal reasoning system which is at least as strong as the standard axiomatization of elementary number theory (the "first order Peano arithmetic").

(VV's remark in the talk: THIS is the correct formulation of the theorem!)

Gödel Paradox:

- We "know" that the first order arithmetic is consistent; (This is commonly accepted fact among mathematicians and as a consequence among everybody else.)
- It can be proved that it is impossible to prove that the first order arithmetic is consistent.

VV's remark: This is extremely unsettling for any rational mind!

Three Choices:

1. If we somehow "know" that the first order arithmetic is consistent than we should be able to transform this knowledge into a proof and then the second incompleteness theorem is false as stated. (Attempted consistency proofs for PA.)
2. Admit a possibility of "transcendental", provably unprovable knowledge.
3. Admit that the feeling of "knowing" in this case is an illusion and that the first order arithmetic is inconsistent (sic!).

VV discards 2) outright. He discards 1) by arguments, which follow. He claims that 3) is the sole rational option. He sees GII is a preliminary step toward the wanted inconsistency proof.

ingnoramus?

VV does not qualify the *ignoramus et ignoramibus* as a rational option.

Cf. Hilbert's 1930 reaction to Gödel Incompleteness :

Wir müssen wissen, wir werden wissen.

The existence of $\mathbb{N}$ as a proof of Con(PA)

Objection (VV): Subsets of $\mathbb{N}$ (or $\mathbb{N}^k$ for a finite k) defined by first-order formulas are, generally, not decidable/recursive/computable (in PRA?).

Gentzen's 1936 of Con(PA)

Objection (VV): is not convincing either because the last step is taken as self-evident while it is actually not (without elaboration)

Dialetheic mathematics?

"The nature of Goedel's argument shows that it is impossible to construct foundations for mathematics which will be provably consistent. "

"What we need are foundations which can be used to construct reliable proofs despite being inconsistent. "

Dialetheic mathematics?

"In mathematics — this is my idea, the only solution which comes to my mind — we may have to learn to use inconsistent systems to obtain reliable proofs. And, ultimately, if we do learn to do such a thing it will be very liberating because then one can use reasoning systems which are known to be inconsistent but which are closer to our intuitive thinking [in order] to construct proofs which then can be verified formally to be reliable. Ultimately this can lead to more freedom in the mathematical workflow. "

Cf. Hilbert on the "axiomatic freedom"!

Dialetheic mathematics?

“The inconsistency of the 1st order arithmetic doesn’t mean that 2+2 doesn’t equal 4. Please don’t worry. [...] It really has nothing to do with the validity of usual computational mathematics, which is being used all around us. If the 1st order arithmetic is inconsistent this doesn’t mean that planes start falling from the air and bridges start falling down. No. The actual computational mathematics is supported by much more than our believe in the consistency of formal theory. ”

Disclaimer:

The talk has been also discussed in other forums and blogs (including MathOverFlow, M-Phi and some other), which I didn't follow.

Neil Tennant:

If a Fields Medallist working in algebraic geometry and homotopy theory is able to give an account of GII at only such an amateurish level, what hope is there for the future of fom in Departments of Mathematics?

me:

Perhaps this future is not so bright indeed (as far as one sticks to a narrow meaning of fom) but I think that at least a part of the problem is that (at least a part of) fom community deliberately isolates itself from the rest of mathematical community. This is harmful for the fom community at the first place.

(Note 2018: Agnus MacIntyre stressed the same point in his invited lecture in KGRC earlier the same year. In this lecture he also praised Voevodsky's 2010 talk and admitted that the inconsistency of PA is a possibility.)

Martin Davis:

I have neither an opinion about nor the least understanding of Voevodsky's technical work. But referring to that work in the context of the silly notion that Gödel's work casts serious doubt on the consistency of PA does not make this silly idea less silly.

Joe Shipman:

The consistency of PA is a THEOREM. It HAS BEEN PROVEN. Full Stop. Although philosophers may dispute this, professional mathematicians may ONLY do so if they publicly criticize Kruskal's theorem, Ulm's theorem, and many much more elementary results as dubious and not worthy of the status of "theorem".

Tim Chow:

[L]et us consider PA, the axioms of first-order arithmetic. Since it is standard mathematical practice to assume [. . .] the existence of N, the only question is whether N satisfies the axioms of PA. The only axiom that could possibly create an issue is the induction axiom. But it is clear that a first-order formula defines a precise property of N, on which we can of course perform induction. So N is indeed an example of something that satisfies the axioms of PA.

(Note 2018: Cf. my exchange with Lev Beklemishev.)

Richard Heck:

Voevodsky means something other by "proof" than what we would normally assume: that he is worried that any proof of Con(PA) must, in some epistemic and not purely mathematical sense be circular, in the sense that such a proof could carry no weight for someone worried about whether PA is actually consistent. So we might well have a proof, in one sense, but not one that satisfies.

William Tait:

I think that Voevodsky is right: whatever researches in proof theory or foundations of mathematics are to accomplish, the search for nontrivial consistency proofs is off the board. This may also be what Andrei Rodin meant when he wrote:

While for the in-consistency of PA and ZFC we may possibly have a sound *mathematical* argument (evidence) any attempted proof of the consistency of these systems will be not a mathematical proof proper but involve some further non-mathematical assumptions.

William Tait (elswhere):

No matter how familiar we become with set theory, or even arithmetic, as at least partially expressed by axioms, and no matter how intuitive those axioms are or become for us, consistency is just something that, ultimately, we must take on faith. And if we should discover a contradiction in Peano Arithmetic, say, that would not show that numbers do not exist: rather it would undermine the sense of existence assertions concerning numbers (and so the sense of their negations, as well (Beyond the axioms: the question of objectivity in mathematics Philosophia Mathematica 9 (2001): 21-36)

VV's "naive" formulation of GII

VV's formulation of GII makes perfect sense (and is not in odds with its more detailed expositions and elaborations) only if one makes the following further epistemic assumption:

A proof of $Con(T)$ that involves theory T', which is stronger than T, is not (epistemically) valid.

VV's "naive" formulation of GII

The above assumption rules out objections (LB, JS and many other people) according to which:

- Since there are proofs of $Con(T)$, which involve no theoretical means beyond those routinely used in Calculus and other areas of the "pedestrian everyday mathematics" ...
- such proofs should be endorsed by any mathematician. (Contrapositively: anyone who rises doubts about such a proof of $Con(T)$ should also rise doubts about a large bunch of standard mathematical results.)

VV's "naive" formulation of GII

The conclusion does not follow because the endorsement of proofs depends on the context: tools which are acceptable in some contexts are not acceptable in some other contexts (for epistemic reason). Standard results, which involve arithmetic in some form, may essentially not depend on whether PA is consistent or not.

VV's "naive" formulation of GII

Unlike most of logicians/proof-theorists VV always speaks of mathematical proof in an "absolute" sense of an evidence for truth.

The proof-theorists like all working mathematicians also use such a notion when they prove and present their results. However when they take "proofs" as objects of their mathematical studies they speak of proof in a different sense. This creates a systematic semantic and conceptual ambiguity. The common proof-theoretic relativism, however sophisticated, is a somewhat natural reaction to Gödel Paradox but hardly its sound solution!

This problem is presently recognised by a part of the proof-theoretic community who develop General Proof Theory (the term introduced by Dag Prawitz back in 1970-ies) and Proof-Theoretic Semantics (Peter Schröder-Heister et al.)

No ignoramus and Dialetheism

VV's "no ignoramus" position w.r.t. foundations (which mirrors Hilbert's view) rejects the popular wisdom according to which we are doomed to be ignorant about such matters because of Gödel Incompleteness. VV finds an original way to stick to the no ignoramus epistemic principle in mathematics at the presence of Gödel Incompleteness.

It makes an interesting link to the Dialetheism in Logic and Epistemology defended by Graham Priest and his followers. Most certainly VV was not aware of these works.

Constructive Validity

A less radical (and somewhat pragmatic) aspect of VV's message is this: Completeness and Consistency, which have been in the main focus of FOM research since 1930-ies, are less pertinent issues in FOM than they are usually taken to be. This wrong focus resulted into an intolerable gap between FOM and (all other parts of) Mathematics.

What the FOM research should focus on in its stead is a notion of *constructive validity* (or perhaps validity *tout court*) of mathematical arguments and mathematical constructions. Such a notion is pertinent for all mathematical research, particularly in the present situation when proof-checking becomes a real problem (cf. eg. the recent case of Michael Atiyah's announced RH proof).

Univalent Foundations

The Univalent Foundation program is an ongoing continuing attempt to build new FOM along the above lines.

It goes without saying that VV and his supporters including Agnus MacIntyre, William Tait and myself conceive of FOM in a different way than do Harvey Friedman, Martin Davis and other people who continue to pursue the mainstream 20th century line in FOM.

A Springer volume on different conceptions of FOM edited by Deborah Kant and Deniz Sarikaya will appear in 2018 (hopefully). The volume is prepared after conference "Foundations of Mathematics: Univalent Foundations and Set Theory" held in July 2016 in Bielefeld, Germany. (Unfortunately Volodya did not have time to write down his talk in a paper form.)

THANK YOU