From Logical Geometry to Geometrical Logic

The Case of Univalent Foundations

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Hilbert 1899: Grundlagen

Geometry [..] requires for its **logical development** only a small number of simple, fundamental principles. These fundamental principles are called the axioms of geometry. The choice of the axioms [..] is tantamount to the **logical analysis** of our intuition of space.

Tarski 1959

What is Elementary Geometry?

[W]e regard as elementary that part of Euclidean geometry which can be formulated and established [in the form of formal first-order theory] without the help of any set-theoretical devices.

Tarski 1936-41

Introduction to the Methodology of Deductive Sciences

When we set out to construct a given discipline, we distinguish, first of all, [..] PRIMITIVE TERMS or UNDEFINED TERMS, and we employ them without explaining their meanings. [..]

We proceed similarly with respect to the asserted statements of the discipline in question. Some of these statements, whose truth appears to us evident, are chosen for the so-called PRIMITIVE STATEMENTS or AXIOMS [..]; we accept them as true without establishing them in any way.

Tarski 1936-41 (continued)

[W]e agree to accept any other statement as true only if we have established its validity, and if we used for this purpose nothing but the axioms, the definitions, and those statements of the discipline which were established previously [..]. [S]tatements which are justified in such a way are called PROVED STATEMENTS or THEOREMS, and the processes of justifying them are called PROOFS.

[Notice that Rules of Inference are not mentioned in this short description!]

Tarski 1936-41 (continued)

Contemporary mathematical logic is one of those disciplines which have been constructed in accordance with the principles just stated [..].

If any other discipline is constructed in accordance with these principles, it is already based upon logic; logic, so to speak, is then already presupposed. This means that all expressions and all laws of logic are treated on an equal footing with the primitive terms and the axioms of the discipline in question [..].

[L]ogic itself does not presuppose any preceding discipline.

Tarski 1936-41 (continued)

Interpretation of formal sentence: assignment of meaning to **variables and non-logical constants (individual and predicate symbols**. The meaning of **logical** constants (if any) is rigidly fixed and does not vary from one interpretation to another.

Recent developments

- Elementary (Logical) Geometry: Victor Pambuccian (Arizona State U) et al.
- Formal (Philosophical) MereoTopology: Varzi, Smith, Arntzenius and many others

Tarski 1938

Sentential Calculus and Topology

Topological interpretation of Classical and Intuitionistic Propositional Calculi.

- $X \vee Y \rightsquigarrow X \cup Y$
- $X \wedge Y \rightsquigarrow X \cap Y$
- $X \rightarrow Y \rightsquigarrow U - Cl(X - Y)$
- $\neg X \rightsquigarrow U - Cl(X)$

Logical Constants are given extra-logical meanings! The Methodology does not apply. Topology is used as a meta-mathematical tool for studying provability in propositional deductive systems.

McKinsey&Tarski 1944

The Algebra of Topology

Topological semantics for modal logic S_4. Idea: interpret $\diamond$ as a closure operator.

Appendix 3: In various discussions of this subject in the literature, one can find quite a different definition of a free algebra with a given number of generations, and also a different proof of the existence of such algebras. Both the definition and the proof use certain terms of a **meta-mathematical** character. Thus for instance a free algebra with n generators is sometimes defined as one in which every equation which holds between generators is a consequence of the postulates defining this algebra [..].

More recent developments

- Topos theory : Lawere at al.: Logic as a part of Geometry;
- Formal Topology, Theory of Locales: Vickers, Sambin at al.
- Modal Logic of Space (Spatial Logics): van Benthem et al.
- Epistemic Logic via Topology: Parikh et al.
- Homotopy Type theory and Univalent Foundations of Mathematics

In fact, the *Geometrical Logic* does serve as an effective means of formalisation of geometrical and other theories.

Logic is a part of Geometry?

Lawvere 1970: [I]n a sense logic is a special case of geometry

The received "Scholastic" Tarski's Methodology is a mistake?

Convergence? Spatial Logics 2005

Spatial logics arise by making a number of design choices, along three principal dimensions.

- The first concerns the collection of geometrical entities which make up our interpretations: points, lines, regions (of various kinds), and so on [..].
- The second principal dimension concerns the choice of primitive relations and operations over these entities to interpret the **non-logical primitives** of our language.
- The third principal dimension concerns the purely logical resources at our disposal. We have already seen that these can be set at many levels: from weak 'constraint' languages through to richer first-order languages or even higher-order formalisms which include the resources of set theory.

Johnstone 1983

The Point of Pointless Topology

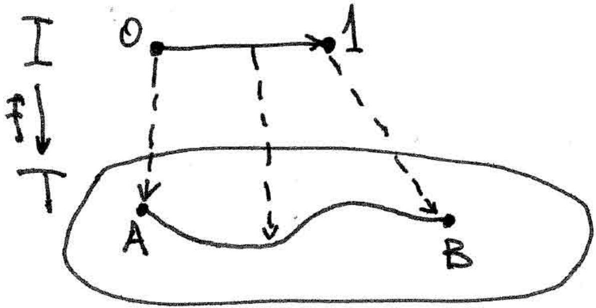
vs.

Whitehead 1919, Tarski 1927 (Geometry of Solids), Menger 1940 and their modern "mereotopological" heirs.

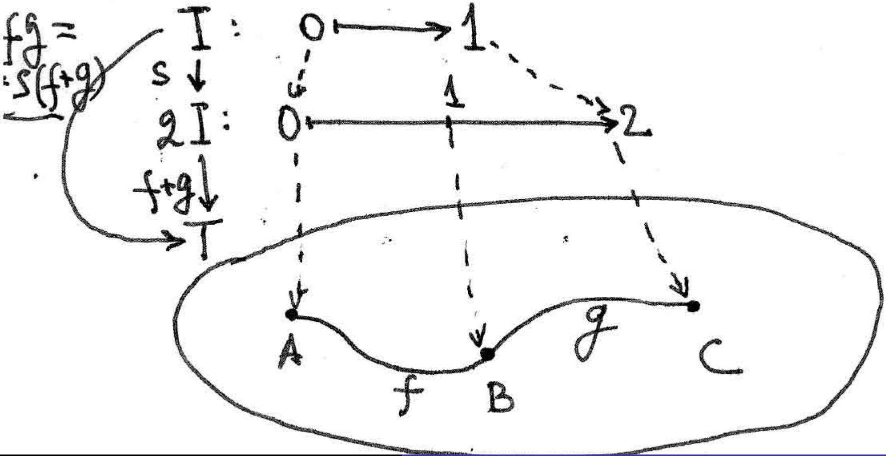
A new Methodology of Deductive Sciences is wanted!

No time for general discussion. But below I explain a novel way of delimiting the bounds of Logic that the UF suggests.

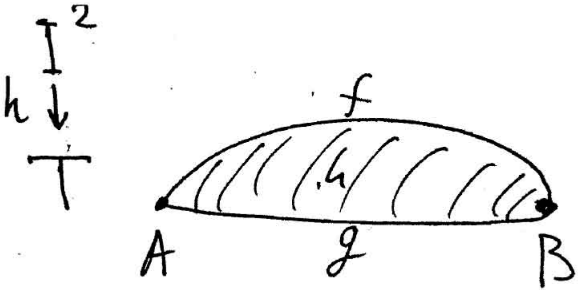
Path



Path Composition



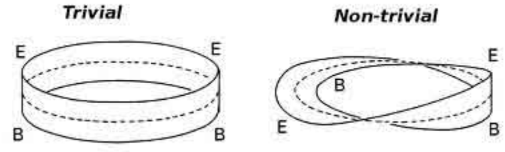
Homotopy Categories and Higher Categories



Fundamental Group and Fundamental Groupoid of Topological Space

2- and higher groupoids

Fibration: trivial and non-trivial



MLTT: Syntax: declarations (judgements)

4 basic forms of judgement (declaration):

(i) $A : TYPE$;

(ii) $A \equiv_{TYPE} B$;

(iii) $a : A$;

(iv) $a \equiv_A a'$

Syntax: Dependent Types and Context

$$\frac{X, A : TYPE; x : X}{A(x) : TYPE}$$

Context Γ is built inductively from declarations respecting type dependencies.

$\Gamma \vdash a : A$
reads: declaration (judgement) $a : A$ is valid (term a of type A is constructible) in context Γ.

Syntax: (some) Rules

Function types (= constant dependent types):

$$\frac{\Gamma \vdash X : TYPE; \Gamma \vdash X : TYPE}{\Gamma \vdash X \rightarrow A : TYPE}$$

$$\frac{\Gamma, x : X \vdash a(x) : A; \Gamma \vdash X : TYPE}{\Gamma \vdash (\lambda x : X)a(x) : X \rightarrow A : TYPE}$$

$$\frac{\Gamma \vdash f : X \rightarrow A; \Delta \vdash x : X}{\Gamma, \Delta \vdash apply(f, x) : A}$$

Syntax: Axioms

NONE

MLTT is Gentzen-style (rule-based) but not Hilbert-style (axiom-based) logical calculus!

Semantics (meaning explanation) of $t : T$ (Martin-Löf 1983)

- t is an element of set T
- t is a proof (construction) of proposition T ("propositions-as-types")
- t is a method of fulfilling (realizing) the intention (expectation) T
- t is a method of solving the problem (doing the task) T (BHK-style semantics)

Sets and Propositions Are the Same

If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [...] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [...] apart. Instead we simply identify them, that is, treat them as one and the same notion. (Martin-Löf 1983)

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x=_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is not a theorem in the (intensional) MLTT (Streicher 1993).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- ...

HoTT: the Idea

Types in MLTT are (informally!) modelled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in Category theory (in which case one thinks of *n*-groupoids as higher homotopy groupoids of an appropriate topological space).

Identity types are modelled by path spaces.

Dependent types are modelled by fibred spaces.

2 Remarks

WARNING!: The above is not a fair Model theory of HoTT but a pre-formal geometrical intuition (cf. the usual informal Euclidian geometry). A true homotopical model of MLTT in the category of Simplicial Sets has been built by Voevodsky some time between 2006 and 2009 and first published by Kapulkin and Lumsdain in 2012: https://arxiv.org/abs/1211.2851 . This advanced material is not found in the HoTT Book.

As far as MLTT qualifies as a logical calculus, broadly conceived, and as far as Homotopy theory qualifies an an (extra-logical) geometrical theory, also broadly conceived, the homotopical interpretation of MLTT (= HoTT) interprets logical syntax with extra-logical geometrical concepts. We shall shortly see how HoTT settles this paradox.

Homotopical interpretation of Intensional MLTT

- $x, y : A$
  x, y are points in space A
- $x', y' : x =_A y$
  x', y' are paths between points x, y; $x =_A y$ is the space of all such paths
- $x'', y'' : x' =_{x =_A y} y'$
  x'', y'' are homotopies between paths x', y'; $x' =_{x =_A y} y'$ is the space of all such homotopies
- ...

Point

Definition

Space S is called contractible or space of h-level (-2) when there is point $p : S$ connected by a path with each point $x : A$ in such a way that all these paths are homotopic (i.e., there exists a homotopy between any two such paths).

Homotopy Levels

Definition

We say that S is a space of h-level $n+1$ if for all its points x, y path spaces $x =_S y$ are of h-level n.

Cummulative Hierarchy of Homotopy Types

- -2-type: single point *pt*;
- -1-type: the empty space ∅ and the point *pt*: truth-values aka (mere) propositions
- 0-type: sets: points in space with no (non-trivial) paths
- 1-type: flat groupoids: points and paths in space with no (non-trivial) homotopies
- 2-type: 2-groupoids: points and paths and homotopies of paths in space with no (non-trivial) 2-homotopies
- ...

Propositions-as-**Some**-Types !

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

Truncation

Each type is transformed into a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

- Thus in HoTT "merely logical" rules (i.e. rules for handling propositions) are instances of more general formal rules, which equally apply to non-propositional types.
- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as *constructive* theory in the sense that besides of propositions it comprises non-propositional objects (on equal footing with propositions rather than "packed into" propositions as usual!) and formal rules for managing such objects (in particular, for constructing new objects from given ones). In fact, HoTT comprises rules with apply *both* to propositional and non-propositional types.

Univalence

$$(A =_{TYPE} B) \simeq (A \simeq B)$$

In words: equivalnce of types is equivalent to their equality.

For PROPs: $(p = q) \leftrightarrow (p \leftrightarrow q)$ (propositional extensionality)

For SETs: Propositions on isomorphic sets are logically equivalent (isomorphism-invariance)

Univalence implies *functional extensionality*: if for all $x \;\; X$ one has $fx =_Y gx$ then $f =_{X \to Y} g$ (the property holds at all h-levels).

Open Problem: the Initiality Conjecture

Build a category of models for MLTT (or its replacement) where the *term model* is the initial object. Solved only for Calculus of Constructions (CoC, after Th. Coquand) by Th. Streicher in 1991. CoC is a small fragment of MLTT. Cf. Lawvere's conception of theory as a "generic model".

Models of HoTT and the Constructive View of Theories, forthcoming in: Stefania Centrone, Deborah Kant and Deniz Sarikaya (eds.) Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts, Springer, Synthese Library. Preprint: http://philsci-archive.pitt.edu/14434/

HoTT/UF as a formal theory

HoTT/UF is an effective formalisation of Homotopy theory, which extends to some neighbouring theories (including Homological Algebra) even if not to all mathematics (but arguably it does). It is a successful case of application of logical methods in mathematics.

HoTT/UF as a formal theory

However HoTT/UF is not a Hilbert-style axiom-based theory built according to Tarski's 1936 Neo-Scholastic Methodology but a Gentzen-style rule-based calculus.

Rules of HoTT/UF applied to propositional (-1) types qualify as logical rules (rules of logical inference); the same rules applied to higher types qualify as rules of geometrical constructions, which prove propositions obtained via the propositional truncation of those types. Thus HoTT/UF can be seen as a formal version of what Hilbert called *genetic* or *constructive* method of theory-building as opposed to his then-new *existential axiomatic* method.

Logicality in HoTT

HoTT/UF does not require the controversial talk about the "interpretation of logical syntax with extra-logical elements' but provides an internal criterion of logicality. It provides a precise sense in which Logic is a part of Geometry. The logical calculus in question has been not designed to support a spatial reasoning specifically; it is not on a par with Temporal Logic. It is an universal logical calculus, which embeds into a geometrical theory. This properly geometrical part of HoTT/UF have a big potential in applications (also beyond the pure Mathematics), which still waits to be explored.

Comparing the HoTT criterion of logicality with other criteria found on the market remains an open research problem.

New Modern Methodology of Deductive Sciences?

The Book of Nature and Human Technology is written in the mathematical language, and its characters are triangles, circles and other homotopy types; without these, one is wandering in a dark labyrinth. Logic is a basic element of this language but not its proper Foundation. Logic alone is incapable to account for the Form of (techno-)scientific reasoning, let alone for its Content.

THANK YOU