Objects & Proofs

Session 2: Sets, Categories, Toposes and the Idea of Internal Logic

26 July

The axioms of Zermelo-Fraenkel set theory with choice

In principle all of mathematics can be derived from these axioms!

Extensionality

$$\forall X \, \forall Y \, [X = Y \quad \Leftrightarrow \quad \forall z (z \in X \; \Leftrightarrow \; z \in Y)]$$

Pairing

$$\forall x\, \forall y\, \exists Z\, \forall z\, [\, z \in Z \quad \Leftrightarrow \quad z = x \text{ or } z = y\,]$$

Union

$$\forall X\, \exists Y\, \forall y\, [\, y \in Y \quad \Leftrightarrow \quad \exists Z (Z \in X \text{ and } y \in Z)\,]$$

Empty set

$$\exists X \, \forall y \, [\, y \notin X \,]$$

(this set X is denoted by $\emptyset$)

Infinity

$$\exists X\,[\,\emptyset \in X \text{ and } \forall x(x \in X \Rightarrow x \cup \{x\} \in X)\,]$$

Power set

$$\forall X\, \exists Y\, \forall Z\, [\, Z \in Y \quad \Leftrightarrow \quad \forall z (z \in Z \;\Rightarrow\; z \in X)\,]$$

Replacement

$$\forall x \in X\, \exists! y\, P(x,y) \quad \Rightarrow \quad [\,\exists Y\, \forall y\, (y \in Y \;\Leftrightarrow\; \exists x \in X\, (P(x,y)))\,]$$

Regularity

$$\forall X\,[\,X \neq \emptyset \quad \Rightarrow \quad \exists Y \in X\,(X \cap Y = \emptyset)\,]$$

]

Axiom of choice

$$\forall X\,[\emptyset \notin X \text{ and } \forall Y, Z \in X (Y \neq Z \;\Rightarrow Y \cap Z = \emptyset)$$

$$\Rightarrow \quad \exists Y\, \forall Z \in X\, \exists! z \in Z\, (z \in Y)\,]$$

Elementary Theory of Category of Sets (ETCS)

Lawvere 1964

The Idea (back to von Neumann in late 1920-ies): use functions and their composition instead of sets and the primitive membership relation $\in$ used in the ZF and its likes

Remark: The project as it stands is fully compatible with RAM; the proposed deviation from the standard approach amounts to a new choice of primitives.

ETCS 1: ETAC

Elementary Theory of Abstract Categories (Eilenberg - MacLane)

- E1) $\Delta_i(\Delta_j(x)) = \Delta_j(x); i,j = 0,1$
- E2) $(\Gamma(x,y;u) \wedge \Gamma(x,y;u')) \Rightarrow u = u'$
- E3) $\exists u \Gamma(x,y;u) \Leftrightarrow \Delta_1(x) = \Delta_0(y)$
- E4) $\Gamma(x,y;u) \Rightarrow (\Delta_0(u) = \Delta_0(x)) \wedge (\Delta_1(u) = \Delta_1(y))$
- E5) $\Gamma(\Delta_0(x),x;x) \wedge \Gamma(x,\Delta_1(x);x)$
- E6) $(\Gamma(x,y;u) \wedge \Gamma(y,z;w) \wedge \Gamma(x,w;f) \wedge \Gamma(u,z;g)) \Rightarrow f = g$

E1)-E4): bookkeeping (syntax); 5): identity; 6): associativity

ETCS 2: Elementary Topos (anachronistically):

- finite limits;
- Cartesian closed (CCC): terminal object (1), binary products, exponentials;
- subobject classifier

$$
\begin{tikzcd}
U \arrow[r, "!"] \arrow[d, hook, "p"'] & 1 \arrow[d, "true"] \\
X \arrow[r, "\chi U"'] & 2
\end{tikzcd}
$$

for all p there exists a unique χU that makes the square into a pullback

ETCS 3: well-pointedness

for all $f, g : A \to B$, if for all $x : 1 \to A$ $xf = xg = y$ then $f = g$

$$\begin{tikzcd}
 & 1 \arrow[dl, "x"'] \arrow[dr, "y"] & \\
A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B
\end{tikzcd}$$

ETCS 4: NNO

Natural Numbers Object: for all t', f there exists unique u

$$\begin{tikzcd}
1 \arrow[r, "t"] \arrow[dr, "t'"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, "u"] & \mathbb{N} \arrow[d, "u"] \\
& A \arrow[r, "f"'] & A
\end{tikzcd}$$

ETCS 4: Axiom of Choice

Every epimorphism splits:
If $f : A \to B$ is epi then there exists mono $g : B \to A$ (called *section*) such that $gf = 1_B$

Cartesian Closedness and the Diagonal Argument

- Lawvere 1969: CCC is a common structure shared by (1) the simply typed λ-calculus (Schönfinkel, Curry, Church) and (2) Hilbert-style (and Natural Deduction style) Deductive Systems (aka Proof Systems).
- In other words CCC is "the" structure captured by the *Curry-Howard correspondence* or *Curry-Howard isomorphism*
- The CCC structure is *internal* for *Set* BUT is more general: *Cat* (of all *small* categories) is another example; any *topos* is CCC.
- Lawvere's Hegelian understanding of this issue: CCC is *objective* while usual syntactic presentations or logical calculi are only *subjective*. While syntactic presentations lay out only *formal* foundations, CCC lays out a *conceptual* foundation.

Internal Logic of Topos more formally

- Syntax (Mitchell-Bénabou): A sorted language L with lists of variables of every sort; sorts correspond to objects of the given topos; logical operations are compatible with usual operations with topos objects (produce, exponentiation, etc.)
- (External) Semantics (Kripke-Joyal): a formal satisfaction relation.

External vs. Internal View

Def. $f : A \to B$ is *epic* iff for all g, h $\quad gf = hf$ implies $g = h$.
Def. Object T is *terminal* if for all object X there is unique arrow $X \to T$
Def. Arrow of the form $e : T \to A$ is called an *element of* A: $e = \in_T A$

Fact: $f : A \to B$ is epic iff it is internally onto: $y.B \vdash (\exists x.A)y = fx$

Warning: $y.B \vdash (\exists x.A)y = fx$ does <u>not</u> say that for each $y \in_T B$ there exists some $x \in_T A$ such that $y = fx$. Externally, f is epic but not necessarily *split* epic.

Prehistory of Internal Logic

- Boole 1847, Venn 1882: propositional logic as algebra and mereology of (sub) classes (of a given universe of discourse); logical diagrams
- Tarski 1938 topological interpretation of Classical and Intuitionistic propositional logic

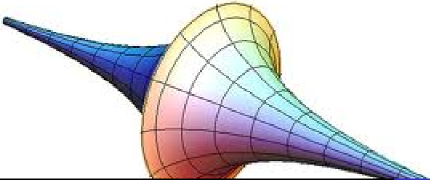
While in Boole, Venn and Tarski an internal treatment is given only the propositional logic Lawvere develops a similar approach to the 1st-order logic. There is a significant technical and conceptual advance between the two cases.

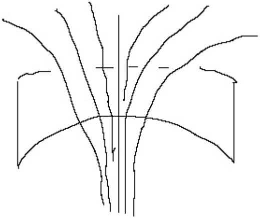
A Geometrical Analogy

Riemann's 1854 notion of Internal Geometry: manifolds as curve spaces.

Beltrami's Pseuso-Sphere vs. Lobachevsky's Hori-Spehre.

Kantian (transcendent), Riemanian (intrinsic, immanent) and Lebnizian-Lobachevskian (relational) conceptions of geometrical space.





Lawvere: Quantifiers and Sheaves 1970

"The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction: a Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for "the" category $\underline{S}$ of abstract sets to an arbitrary topos.

Lawvere: Quantifiers and Sheaves 1970

We first sum up the principle contradictions of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors [..] enabling one to claim that in a sense logic is a special case of geometry.

Lawvere: Quantifiers and Sheaves 1970

"When the main contradictions of a thing have been found, the scientific procedure is to summarize them in slogans which one then constantly uses as an ideological weapon for the further development and transformation of the thing. Doing this for "set theory" requires taking account of the experience that the main pairs of opposing tendencies in mathematics take the form of adjoint functors and frees us of the mathematically irrelevant traces ($\in$) left behind by the process of accumulating ($\cup$) the power set (P) at each stage of the metaphysical "construction". Further, experience with sheaves, [..], etc., shows that a "set theory" for geometry should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters."

"What is to be considered is the whole Notion, firstly as the Notion in the form of being, secondly, as the Notion; in the first case, the Notion is only *in* itself, the Notion of reality or being; in the second case, it is the Notion as such, the Notion existing *for* itself (as it is, to name concrete forms, in thinking man, and even in the sentient animal and in organic individuality generally [..]). Accordingly, logic should be divided primarily into the logic of the Notion as being and of the Notion as Notion - or, by employing the usual terms (although these as least definite are most ambiguous) into 'objective' and 'subjective' logic. " (BL)

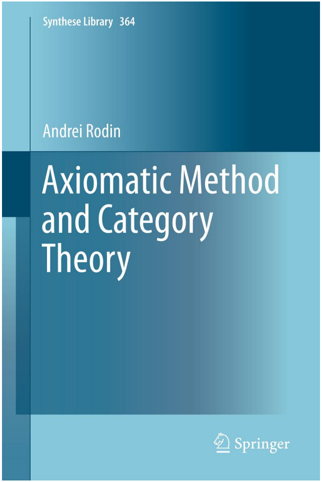
Lawvere: Categories of Space and Quantity 1992

It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy.

Badiou

Topos Intuistionistic logic as a transcendental logic over the “true” Classical Logic of (the category) Sets.

Book


Synthese Library 364
Andrei Rodin
Axiomatic Method and Category Theory
Springer

Genetic method: Hilbert 1900

The idea: Mathematical objects are built from other such objects. More complex objects are built from simpler ones.

Hilbert's example

Dedekind Cuts and Cochy sequences. Both are "built from" natural numbers.

Notice that neither of the two "constructions" is constructive in any the usual senses of the word (Turing, Bishop, Markov, et. al)!

Genetic method: Hilbert 1900

Despite the high pedagogic and heuristic value of the genetic method, for the final presentation and the complete logical grounding of our knowledge the axiomatic method deserves the first rank.

Hilbert & Bernays 1934

The term axiomatic will be used partly in a broader and partly in a narrower sense. We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry, Newton for mechanics, and Clausius for thermodynamics. [..].

Hilbert & Bernays 1934

[F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* [in the narrow sense?] and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory only as a *genus* of things, an axiomatic theory refers to a fixed system of things [..] given as a whole. Except for the trivial cases where the theory deals only with a finite and fixed set of things, this is an idealizing assumption that properly augments the assumptions formulated in the axioms.

Hilbert & Bernays 1934

Euclid does not presuppose that points or lines constitute any fixed domain of individuals. Therefore, he does not state any existence axioms either, but only construction postulates. (op. cit. p. 20a)

Objectual Constructivity

A constructive axiomatic theory, generally, comprises extra-logical rules similar to Euclid's Postulates, which allow for building of and operating with non-propositional objects. While usual notions of constructivity in logic and mathematics specify such rules in one way or another I leave it wholly open *what* such rules may or should be.

The theory of *Elements*, Book 1 qualifies as constructive in *that* sense. As we shall see HoTT does so too.

THANK YOU