Objects & Proofs

Session 3: Axiomatic Method, Objecthood and the Univalent Foundations of Mathematics

27 July

Ryle 1945 on Knowing How to Reason Logically

[T]he intelligent reasoner is knowing rules of inference whenever he reasons intelligently'. Yes, of course he is, but knowing such a rule is not a case of knowing an extra fact or truth ; it is knowing how to move from acknowledging some facts to acknowledging others. Knowing a rule of inference is not possessing a bit of extra information but being able to perform an intelligent operation. *Knowing a rule is knowing how.* (The emphasis is added by the authors.)

Hilbert 1894

Among the appearances or facts of experience manifest to us in the observation of nature, there is a peculiar type, namely, those facts concerning the outer shape of things, Geometry deals with these facts [..]. Geometry is a science whose essentials are developed to such a degree, that all its facts can already be *logically deduced* from earlier ones.

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- If different points A, B belong to straight line a and to straight line b then a is identical to b.

The distinction between logical and non-logical concepts plays a fundamental role in Hilbert's 1899 axiomatics because it provides a sense of being formal for his axiomatic theories. The form in point is a logical form. That means that logical semantics (which is not explicitly construed in this framework!) is rigidly fixed and the non-logical semantics is left variable. A formal mathematical theory is grounded in logic and logic alone. Logic in this context is thought of as a system of rules for handling propositions.

Hintikka 1997 on Hilbert-style axiomatic method

The basic clarified form of mathematical theorizing is a purely logical axiom system.

Rules versus Axioms

"The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. "(Gentzen 1935)

Natural Deduction and Sequent Calculus

Gentzen 1935: semantics via introduction rules

The introductions [i.e. introduction rules] represent, as it were, the 'definitions' of the symbol concerned.

General Proof Theory

In model theory, one concentrates on questions like what sentences are logically valid and what sentences follow logically from other sentences. But one disregards questions concerning how we *know* that a sentence is logically valid or follows logically from another sentence. General proof theory would thus be an attempt to supplement model theory by studying also the evidence or the process - i.e., in other words, the proofs - by which we come to know logical validities and logical consequences. (Prawitz 1974)

Meaning Explanation

is analogous to program compiler: translation of the given syntax into elementary self-evident steps of reasoning (Martin-Löf 1985)

Proof-theoretic Semantics

PTS is not denotational. It does not assign certain entities to certain symbols. It assigns to symbols (and first of all to logical symbols, i.e., to logical constants) their *meaning*, which is not construed in this case as an entity. The procedure of such an assignment is called after Martin-Löf 1985 the *meaning explanation* and consists, roughly, of the explication of computational content of logical constructions in terms of their building blocks, which are presented in a self-explanatory canonical form.

Terminological Issues

Is it justified to call the Gentzen-style formal approach *axiomatic* even if it is rule-based rather than axiom-based?

I tend to answer in positive having in mind Arsitotle's use of the term "axiom". Axiomatic method is more general than we learn it from Hilbert.

Gentzen-style beyond Logic?

Until recently the Gentzen-style rule-based formal approach has been used only in purely logical calculi (with the only exception of formal arithmetic). Unlike Hilbert Gentzen never attempted to use his version of axiomatic method in all areas of mathematics or in sciences. Univalent Foundations is the first attempt to use this approach more widely in mathematics. I suggest that it can be also effectively used in Knowledge Representation in general.

This dimension of the "axiomatic freedom" waits to be explored!

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- Allow for managing proofs of very different kinds, which include both mathematical arguments and empirical evidences;
- Support the representation of all sorts of *methods* including extra-logical mathematical methods, experimental methods, etc.;
- Are computer-friendly, i.e., allow for encoding into suitable program languages.

Cassirer 1907 adversus Russell 1903

(a teaser for what follows in the remaining lectures)

"Here rises a problem that lies wholly outside the scope of "logistics" [= Formal Symbolic Logic]. All empirical judgements [..] must respect the limits of experience. What logistics develops is a system of hypothetical assumptions about which we cannot know, whether they are actually established in experience or whether they allow for some immediate or non-immediate concrete application. According to Russell even the general notion of magnitude does not belong to the domain of pure mathematics and logic but has an empirical element, which can be grasped only through a sensual perception. From the standpoint of logistics the task of thought ends when it manages to establish a strict deductive link between all its constructions and productions."

Cassirer 1907 adversus Russell 1903

"Thus the worry about laws governing the **world of objects** is left wholly to the direct observation, which alone, within its proper very narrow limits, is supposed to tell us whether we find here certain rules or a pure chaos. [According to Russell] logic and mathematics deal only with the order of concepts and should not care about the order or disorder of objects. As long as one follows this line of conceptual analysis the empirical entity always escapes one's rational understanding. The more mathematical deduction demonstrates us its virtue and its power, the less we can understand the crucial role of deduction in the theoretical natural sciences. " (E. Cassirer, Kant und die moderne Mathematik, 1907)

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- Context : $\Gamma \vdash$ judgement (of one of the above forms)
- no axioms (!)
- rules for contextual judgements; Ex.: dependent product :
  If $\Gamma, x : X \vdash A(x) : TYPE$, then $\Gamma \vdash (\Pi x : X)A(x) : TYPE$

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- t is a method of solving the problem (doing the task) T (BHK-style semantics)

Sets and Propositions Are the Same

If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [. . .] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [. . .] apart. Instead we simply identify them, that is, treat them as one and the same notion. (Martin-Löf 1983)

MLTT: Definitional aka judgmental equality/identity

$x, y : A$ (in words: x, y are of type A)

$x \equiv_A y$ (in words: x is y by definition)

MLTT: Propositional equality/identity

$p : x =_A y$ (in words: x, y are (propositionally) equal as this is evidenced by proof p)

Definitional eq. entails Propositional eq.

$$\frac{x \equiv_A y}{p : x =_A y}$$

where $p \equiv_{x =_A y} refl_x$ is built canonically

Equality Reflection Rule (ER)

$$\frac{p : x =_A y}{x \equiv_A y}$$

ER is <u>not</u> a theorem in the (intensional) MLTT (Streicher 1993).

Extension and Intension in MLTT

- MLTT + ER is called *extensional* MLTT
- MLTT w/out ER is called *intensional*
  (notice that according to this definition intensionality is a negative property!)

Higher Identity Types

- $x', y' : x =_A y$
- $x'', y'' : x' =_{x =_A y} y'$
- $\ldots$

HoTT: the Idea

Types in MLTT are (informally!) modeled by spaces (up to homotopy equivalence) in Homotopy theory, or equivalently, by higher-dimensional groupoids in Category theory (in which case one thinks of n-groupoids as higher homotopy groupoids of an appropriate topological space).

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- $x'', y'' : x' =_{x =_A y} y'$
  x'', y'' <5-> are homotopies between paths x', y'; $x' =_{x =_A y} y'$ is the space of all such homotopies
- ...

Point

Definition

Space S is called contractible or space of h-level (-2) when there is point $p : S$ connected by a path with each point $x : A$ in such a way that all these paths are homotopic (i.e., there exists a homotopy between any two such paths).

Homotopy Levels

Definition

We say that S is a space of h-level $n+1$ if for all its points x, y path spaces $x \; =_S \; y$ are of h-level n.

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Propositions-as-**Some**-Types !

Which types are propositions?

Def.: Type P is a *mere proposition* if $x, y : P$ implies $x = y$ (definitionally).

Truncation

Each type is transformed into a (mere) proposition when one ceases to distinguish between its terms, i.e., *truncates* its higher-order homotopical structure.

Interpretation: Truncation reduces the higher-order structure to a single element, which is **truth-value**: for any non-empty type this value is **true** and for an empty type it is **false**.
The reduced structure is the structure of **proofs** of the corresponding proposition.
To treat a type as a proposition is to ask whether or not this type is instantiated without asking for more.

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- These general rules work as rules of building models of the given theory from certain basic elements which interpret primitive terms (= basic types) of this given theory.
- Thus HoTT qualify as *constructive* theory in the sense that besides of propositions it comprises non-propositional objects (on equal footing with propositions rather than "packed into" propositions as usual!) and formal rules for managing such objects (in particular, for constructing new objects from given ones). In fact, HoTT comprises rules with apply *both* to propositional and non-propositional types.

Syntactic and Semantic (aka Non-Statement) Views on Theories

Syntactic View: A direct Hilbert-style axiomatization of Physical and other scientific theories (since 1900: Hilbert, Rudolf Carnap, Carl Gustav "Peter" Hempel and Ernest Nagel)

Semantic View: A typical scientific theory should be identified with a class with (set-theoretic) models rather than with a particular axiomatic presentation in a formal language (since late 1950-ies: Evert Beth, Patrick Suppes, Bas van Fraassen)

Problem:

None of the above two approaches support an adequate representation of scientific *methods* including methods of justification of scientific claims. This concerns both logical and (particularly) extra-logical methods such as methods of conducting observations and staging experiments.

Non-Propositional Content in Science

HoTT supports a strong version of Non-Sentence View of Theories by providing a precise sense in which a theory, generally, does not reduce to the set of its propositions.

HoTT also supports a *Constructive View* of theories according to which the non-propositional *Knowldge How* is a part of scientific and technical knowledge, which is at least as much important as the propositional *Knowledge That*). HoTT provides a model of how the two sorts of knowledge relate to each other.

Mathematical Modeling in Science

[E]xperience with sheaves, [..], etc., shows that a “set theory” for geometry should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters. (Lawvere 1970 inspired by Hegel)

Logical and mathematical concepts must no longer produce instruments for building a metaphysical “world of thought”: their proper function and their proper application is only within the empirical science. (Cassirer 1907)

Suppes' Lesson

A formal representational framework for Science and Technology should include a formal semantic part rather than apply syntactic structures to material contents directly.

Suppes and his followers use Set theory for that purpose with a relatively little success — at least if this success is measured by the role of formal approaches in the mainstream scientific research. The homotopical semantics can be more appropriate of the task.

Open Problem

It appears that we still miss a good replacement of Tarski's notion of model, which could work with HoTT and CAM more generally. Tarski's notion of satisfaction in its original does not make the whole job in such a context because it involves the concept of truth-evaluation and no alternative notion of model is universally accepted.

The Model theory of HoTT is presently a subject of active research. This research revises basic conceptual issues such as the concept of model itself.

Conclusion 1

The constructive axiomatic architecture is rooted in history (Euclid) as well as in the recent successful practice of axiomatizing geometrical theories (ET, HoTT).

Conclusion 2

As the examples of ET and HoTT clearly demonstrate CAM involves a pattern of relationships between Logic and Geometry, which is quite unlike the corresponding pattern used in RAM. RAM-based axiomatic architecture leaves no room for a conceptual linking of geometrical principles to logical ones. Geometrical axioms appear here as very specific formal principles put on the top of logical principles and motivated solely by unspecified references to spatial experiences and intuitions. The CAM-based axiomatic architecture, in its turn, presents geometrical principles as a generalization of logical principles: in a CAM-based geometrical theory such as HoTT "logic is a special case of geometry".

Conclusion 3

RAM proved effective as a very specific representational tool for meta-mathematical studies. But it appeared to be nearly useless for more general epistemic purposes, for which this method was originally designed or tentatively applied later. This includes the formal proof-checking, developing formal standards in scientific communication and education, developing a software for computer-based Knowledge Representation. Today's science and mathematics applies little of RAM-based methods and of logical methods more generally. Even in CS and software engineering the role of logical approaches appears rather modest.

Conclusion 3 (continued)

CAM already has a better performance and a better record in this respect. Its traditional informal version proved effective in mathematics (Euclid) and physics (Newton, Clausius). Today's proof-assistances such as COQ are CAM-based rather than RAM-based. There are reasons to expect that CAM-based logical methods (and perhaps HoTT more specifically) will apply in today's science and technology (including IT) more effectively that the standard RAM-based methods. In any event it is worth trying.

THANK YOU