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Oxford Computer Lab, June 6, 2008

Euclid, Hilbert and Functorial Semantics

Plan:
1) Euclid: Postulates and Axioms, Problems and Theorems;

2) Standard framework for theory-building: Hilbertian scheme. Frege-Hilbert controversy.

3) Critical arguments against Hilbertian scheme: From sets and structures to categories and functors.

4) Functorial semantics. Cnanging Lawvere's views on foundations.

5) Sketch theory and the method of “generic figures”: back to Euclid?

6) Consclusion

1)Euclid's *Elements*

Is the theory of Euclid's *Elements* a deductive theory (to be improved)?

YES, if any general method of obtaining further theoretical content from assumed first principles qualifies as deduction;

So understood deduction is the same as *generation* according to some rules from a given set of generators.

NO, if deduction is understood as logical inference from axioms.

Different sorts of first principles in *Elements*: Axioms and Postulates (and Definitions, which I don't consider today)

(hereafter tr. Richard Fitzpatrick 2007)

Postulates:

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any centre and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).

Postulates are NOT first *truths*; they are NOT *propositions* in Frege's sense: they do NOT have truth-values. Notice the infinitive form of verbs. "Deduction" from Postulates (whatever this may mean) is NOT logical (truth-preserving) inference. Dubious cases: 4 and 5 (Proclus)

Common Notions [=Axioms: Aristotle]
1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

Unlike Postulates Common Notions (Axioms) *are* first truths; they are proposition in Frege's sense (notice the copula); they can be used as premises in proofs. Axioms hinge upon the notion of *equality*.

Proclus, *Commentary on First Book of Euclid's* Elements, on distinction between Axioms and Postulates:

"What axioms and postulates share in common is the fact that they don't require proofs <...> and serve as foundations for what follows. **Their difference is the same as the difference between theorems and problems.** <...> A postulate requires to invent and arrange some simple matter while an axiom states some essential property well known to listeners."

On Problems and Theorems:

"Problems are propositions where something, which was not earlier given, is built, arranged and exposed while theorems are propositions where properties belonging or non-belonging to a subject-matter under consideration are learnt and proved. In a problem one is supposed to produce, put, apply, inscribe, describe, insert, touch, etc.. while in a theorem one should connect and bind with the proof some properties belonging to geometrical matter."

"A theorem is to be performed as an assertion, for example "two sides of triangle are greater than the third" or "angles at the base of isosceles triangle are equal", while a problem is to be performed as a question, for example "is it possible to construct a triangle on a given side?""

Postulates are first principles for Problems while axioms are first principles for theorems? Not really. For the two things are interwoven: Problems and Theorems share a common structure. This is a reason why Euclid doesn't distinguish between them explicitly.

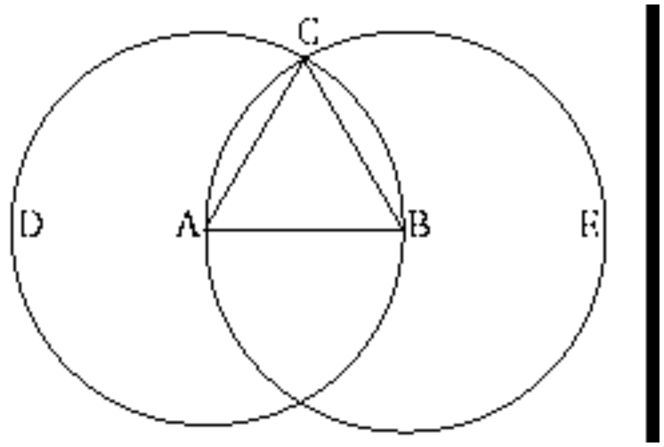
"Every complete problem and theorem should have the following parts:

[1] proposition,
[2] exposition,
[3] limitation,
[4] construction,
[5] proof,
[6] conclusion.

Proposition says what is given and what is wanted. <...> *Exposition* takes the given and prepares it for the search. *Limitation* separates the wanted and clarifies what it is. *Construction* adds to the given what it lacks and prepares the search of the wanted. *Proof* brings together what is present on the basis of assumed premises. *Conclusion* returns to proposition and confirms what is to be shown."

Example 1 (a problem): Proposition 1

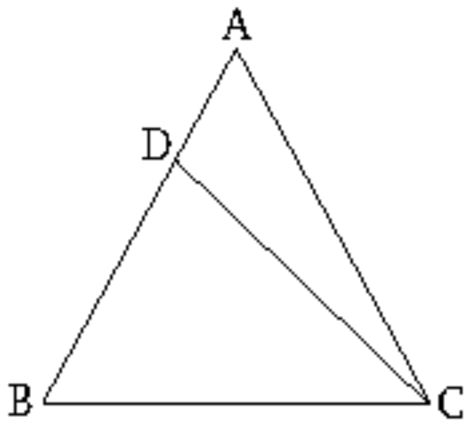
- "To construct an equilateral triangle on a given finite straight-line.
- Let AB be the given finite straight-line.
- So it is required to construct an equilateral triangle on the straight-line AB.
- Let the circle BCD with centre A and radius AB have been drawn [Post. 3], and again let the circle ACE with centre B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B (respectively) [Post. 1].



- And since the point A is the centre of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the centre of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight lines) CA, AB, and BC are equal to one another.
- Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do."

Example 2 (a theorem): Proposition 6

- "If a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another.
- Let ABC be a triangle having the angle ABC equal to the angle ACB.
- I say that side AB is also equal to side AC.



- For if AB is unequal to AC then one of them is greater. Let AB be greater. And let DB, equal to the lesser AC, have been cut off from the greater AB [Prop. 1.3]. And let DC have been joined [Post. 1].
- Therefore, since DB is equal to AC, and BC (is) common, the two sides DB, BC are equal to the two sides AC, CB, respectively, and the angle DBC is equal to the angle ACB. Thus, the base DC is equal to the base AB, and the triangle DBC will be equal to the triangle ACB [Prop. 1.4], the lesser to the greater. The very notion (is) absurd [C.N. 5]. Thus, AB is not unequal to AC. Thus, (it is) equal.
- Thus, if a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another. (Which is) the very thing it was required to show. "

Notice that *proof* is only one element of problem/theorem among 5 others....

Beware of difference between *deixis* and *apodeixis* in Greek; cf. verbs *montrer* and *demontrer* in French or *show* and *prove* in English. Aristotle's Syllogistics concerns apodeixis but not the whole of deixis.

Can Postulates be *interpreted* as existential propositions? In any event the resulting theory will differ significantly from Euclid's. Proclus' Platonic interpretation of *Elements* according to which Postulates are first principles of mathematical Becoming while Axioms are first principles of mathematical Being applies more smoothly. It assumes Platonic distinction btw Being and Becoming and Platonic notion of mathematics as *intermediate* between sensibilia and pure ideas.

Aristotle's notion of science as a body of proposition obtained through logical inference from axioms provides a poor grasp of (his contemporary) mathematics.

The theory of *Elements* combines two different principles of generation of new propositions from the first principles; one for Postulates and the other for Axioms. The latter works only for proofs in the restricted sense. The two are deeply interwoven.

2) Standard view: Hilbertian scheme
Frege's notion of theory is similar to Aristotle's (albeit his logic is different): a body of propositions derived logically from certain axioms. No distinction btw axioms and postulates. In particular this applies to Geometry as a science of space. Geometrical axioms are justified by intuition and validity of inferences is justified by rules of Logic. (The case of Arithmetic, according to Frege, is essentially different since Arithmetic is just as

universal as Logic and arguably makes a part of it.) Truthfulness of geometrical axioms in Frege's view guarantees consistency of Geometry.

Frege, *On foundations of geometry* (a critical review of Hilbert's *Grundlagen der Geometrie*):

"[W]hat is called an axiom is a thought whose truth is certain without, however, being provable by a chain of logical inferences. Logical laws, too, are of this nature. <...> Axioms do not contradict one another, since they are true."

Frege rejects Hilbert's notions of "formal" theory and of *interpretation* of formal theory arguing that unless the *meaning* of a given mathematical statement is fixed it doesn’t have any definite truth-value and hence doesn’t count as a proposition (and in particular cannot count as an axiom).

Here is Hilbert's reply:

"You say that my concepts, e.g. "point", "between", are not unequivocally fixed <...>. But surely it is self-evident that every theory is merely a framework or schema of concepts together with their necessary relations to one another, and that basic elements can be construed as one pleases. If I think of my points as some system or other of things, e.g. the system of love, of law, or of chimney sweeps <...> and then conceive of all my axioms as relations between these things, then my theorems, e.g. the Pythagorean one, will hold of these things as well. In other words, each and every theory can always be applied to infinitely many systems of basic elements. For one merely has to apply a **univocal and reversible one-to-one transformation** and stipulate that the axioms for the transformed things be correspondingly similar. Indeed this is frequently applied, for example in the principle of duality, etc."

Notice a common point of Frege and Hilbert: weak logicism (mathematical propositions are obtained through pure logical deduction from axioms). Frege's approach doesn't work. Formalism is the price for the weak logicism. The "contentual" mathematics is pushed out into models.

From **sets** as "systems of things" to Axiomatic Set theory.

Metatheory of sets.

Carnap 1947, *Formalization of Logic*: syntax and semantics

Model theory

3) From structures to functors, from sets to categories

Why Hilbert's talk of "scheme", "form" and "interpretation"? Standard answer: recognition of *structures*. Think of Euclid's theory of proportions developed separately for *magnitudes* (Books 5-6 of *Elements*) and for *numbers* (Books 7-9) and the lack of any account of their obvious similarity.
In Greek mathematics this account is pushed out into a "universal mathematics", which is a metatheory *about* mathematics. Proclus, *Commentary on Euclid*:

"One shouldn't think after Erathosthenes that Proportion unifies Mathematics. For Proportion is only one thing shared in common by mathematical sciences. The mathematical sciences have many other features belonging to their common nature. Mathematical sciences are unified by the one indivisible Mathematics, which grasps foundations of all particular sciences in their simplest form and considers their differences. <...> However yet at a higher level mathematical sciences are unified by Dialectics."

Notice, however, that there is NO "univocal and **reversible** one-to-one transformation" (Hilbert) between numbers and magnitudes required by Hilbertian scheme...

Categoricity Problem: Hilbertian scheme assumes that possible models of a given formal theory are **isomorphic**. But generally they are not. Hence the pursuit of categoricity. When it doesn't work (like in case of ZF) people often appeal to the notion of "standard" or "intended" model, which has no precise mathematical meaning. So intuitive considerations strike back! Hilbertian scheme doesn't work as it is supposed to.

A deeper reason of Categoricity Problem: Hilbert has two very different notions of interpretation in mind. *First*, he thinks of interpretation of a given formal theory as an appropriate intuitive content, which can be associated with it. This is a philosophical, psychological and pedagogical issue but not a mathematical one. (Do different people imagine Euclidean circles differently?) *Second*, he thinks about a model M of a given formal theory T as a specific construction made within *another* theory T' (supplied by some working model M'). Hilbert's non-trivial mathematical examples are of this second kind. Think of arithmetical models of geometrical theories mentioned in Hilbert's *Grundlagen*.

Claim 1: There is no sufficient reason to treat both notions of interpretation on equal footing. This is a confusion of two very different things.

Argument:
I leave now the issue of intuition aside. But the *second* kind of interpretation can be better understood as a *translation* (*map*, morphism) between theories *T* and *T'* , i.e. interpretation of the theoretical content of *T* in terms of *T'.* This revised notion of interpretation (=translation) cannot be extended to the case of intuitive content (Hilbert's *first* kind of interpretation) because the intuitive content alone doesn't form anything like a theory.

Claim 2:
Hilbertian distinction between mathematics and meta-mathematics is not justified.

Argument:
The usual way to treat translation *T*-->*T'* as interpretation in the *first* (intuitive) sense - to qualify deliberately *T'* as a **meta-**theory and on this ground to leave it out of mathematical consideration - in certain cases

it leads to sheer epistemic absurdities (cf. Lobachevsky's "non-standard" model of Plane Euclidean geometry).

Claim 3:
Mathematically significant translations (maps, morphisms) between theories are generally non-reversible, i.e. not isomorphisms.

Argument:
Otherwise, according to Hilbertian criteria, they are auto-translations of a given theory into itself. Non-trivial reversible auto-translations exist (cf. Hilbert's example of Projective Duality) but are rare. One shouldn’t generalise upon this Hilbert's example.

Remark:
Talking about arithmetical models of geometrical theories Hilbert, of course, didn't mean to identify Geometry with Arithmetic. But he thought he could "carve out" a specific arithmetical construction from its

ambient theory and consider it (with appropriate arithmetical laws) as a self-standing embodiment of a geometrical theory. This is not justified. The construction cannot survive outside its proper theoretical framework.

Claim 4:
Hilbertian scheme doesn't survive the replacement of isomorphisms by general morphisms.

Argument:
Given reversible map *A*<-->*B* one can think of *A*, *B* "up to isomorphism" and identify both *A*, *B* with a new "abstract" or "formal" object *C*. So differences between *A* and *B* can be dispenced with. This is possible because the existence of isomorphism is an equivalence relation, and *C* stands for a particular equivalence class by this relation. But the existence of general morphism *A*-->*B* is NOT an equivalence relation, so nothing similar

applies to the general case. Given general morphism *A-->B* there is no sense in which the difference between *A* and *B* might not matter; there is no way to stipulate in this situation a new "formal" object C like in the special case of isomorphism (or in some similar way).

Remark:
Hilbertian Structuralist setting allows for a rigorous definition and treatment of the general notion of morphism. I mean the structuralist notion of morphism as a structure-preserving map. However this framework is based on a "preference" of isomorphisms to begin with. For the very notion of structure requires the kind of thinking exemplified by the above quote from Hilbert's letter to Frege. Thinking about morphisms as structure-preserving is misleading.

Claim 5:
Set theory is a natural framework for applications of Hilbertian scheme (or a part of it).

Argument (hint):
Any correspondence between two given elements of two given sets is (intuitively) reversible. In Set theory the notion of non-ordered pair is primitive but the notion of ordered pair is derived (construed). In this sense non-reversible correspondences between sets (i.e. functions) and maps between "structured sets" are accounted for in terms of elementary isos (i.e. pointwise).

Claim 5:
Category theory as a general theory of maps is a natural framework for the generalisation of Hilbertian scheme I'm pointing to.

Argument:
Presently we don't have any other proposal.

4**) Functorial semantics; Lawvere on foundations.**

Functorial Semantics of Algebraic Theories" (Thesis of 1963, Author's Commentary of 2004)
Elementary Theory of the Category of Sets, (1964)
The Category of Categories as a Foundation for Mathematics (1966)
Foundations and Applications: Axiomatization and Education (2003)

"Old" (non-Set-theoretic) Hilbertian approach:

"By a category we of course understand (intuitively) any structure which is an interpretation of the elementary theory of abstract categories ..." (*Elementary Theory*)

Functions instead of epsilon:

"There is essentially only one category which satisfies these ... axioms ... , namely the category S of sets and mappings." (*Elementary Theory*)

Here the equivalence of categories replaces isomorphism but I don't see this change as significant.

The idea of functorial semantics: models are functors from a "syntactic" category presenting a (formal) theory to the category of sets.

Hilbertain (Tarskian) feature:
the idea of evaluation of a formal theory in sets (building models out of sets).

New features:

1) categorical syntax:

"We identify objects with their identity maps and we regard a diagram as a formula which asserts that *A* is the (identity map of the) domain of *f* and that *B* is the (identity map of the) codomain of *f*." (*Functorial Semantics*)

2) the requirement of categoricity doesn't make sense and is given up;

3) logic is taken from the "background" (internal logic)

" In the case of logical theories of all sorts the most basic structure they support is an operation of substitution, which is most effectively viewed as a form of composition. Thus, if we construe theories as categories, models are functors! <...> But only for the simplest theories are all functors models,

because something more than substitution needs to be preserved; again, miraculously, the additional features of background categories which were often expressible in terms of composition alone via universal mapping properties, turned out to have precise analogs: the operations of disjunction, existential quantification, etc. on
a theory are all uniquely determined by the behaviour of substitution. Roughly, any collection *K* of universal properties of the category of sets specifies a doctrine: the theories in the doctrine are all the categories having the properties *K*; the mutual interpretations and models in the doctrine are just all functors preserving the properties *K*." (*Commentary*)

3) blurring of the Hilbertian distinction between theories and their models; a given theory "turnes into" one of its models, namely into a "generic" one:

"[T]here was the choice, which I now view as anachronistic, of considering that an algebraic theory is a category with coproducts rather than with products. The "coproduct" convention, which involves defining algebras themselves as contravariant functors from the theory into the background, indeed did permit viewing **the theory itself as a subcategory of the category of models**. However, for logics more general than the equational one considered here, such a direct inclusion of a theory into its category of models cannot be expected. The “product” convention permits the concrete definition of models as covariant functors from the theory; thus the **theory appears itself as a generic model**."
(*Commentary*)

At least a part of these non-Hilbertian features explicitly appears only in *Commentary* of 2004.

In *Foundations and Applications* Lawvere argues for Foundations of Mathematics understood

"... in a common-sense way rather than in the speculative way of the Bolzano-Frege-Peano-Russell tradition".

5) Sketch theory: Back to Euclid?

Generic figures and their gluings by Reyes&Reyes (2004):

Ex: Generation of category of graphs:

$$\boldsymbol{C}\!: V \underset{s}{\overset{t}{\rightrightarrows}} A \;;\; \text{GRAPHS} = (\boldsymbol{C}, \text{SET})$$

Hom (_, V) and Hom (_, A) - representables

Yoneda embedding ***C* -->**(***C***, SET)

Generic figures for GRAPHS are points (vertexes) and arrows.

Remind that GRAPHS has internal logic, so it is a genuine *theoretical* framework.

Sketch theory

Ch. Ehresmann, *Esquisses et types de structures algebriques* (1968)
R. Guitart, *On the Geometry of Computations* (1986)
M. Barr and Ch. Wells, *Category theory for Computing Science* (1990), ch. 7&9.

Observation:
Given a small category ***C*** functor category (***C***, SET) is a *topos*, and hence has "logical properties" (internal logic). This allows for constructions similar to Functorial Semantics without assuming logical properties in ***C***. "Distinguished limits" (=axioms written in

categorical syntax) are replaced by distinguished *cones*; the requirement of *preservation* of the distinguished limits (cf. Lawvere's *doctrines*) is replaced by the requirement that the distinguished cones *turn into* appropriate limits in (***C***, SET).

Pedagogical/CS definitions (Barr&Wells):
1a) A *linear* sketch *S* is a pair (*G*,*D*) where *G* is a graph and *D* is a set of diagrams in *G* (a diagram is a graph homomorphism *I* --> *G*)
1b) A model of a linear sketch *S* in a category *C* is a graph homomorphism *m* : *G* -->*C* such that whenever *d* : *I* --> *G* is a diagram in *D*, then *md* is a *commutative* diagram in *C*.

2a) A *finite discrete* sketch (*G*, *D*, *L*, *K*) where *G* is a finite graph, *D* a finite set of finite diagrams, *L* a finite set of finite *discrete cones* in *G* and *K* a finite discrete set of finite *discrete cocones*
2b) A model of a finite discrete sketch in a category *C* is a model *m* of linear sketch (*G*,

D) with an additional property that for any discrete cone *l* : *i* --> *G* in *L* the composite *ml* is a product cone and for any discrete cocone *k* : *j* --> *G* in *K* the composite *mk* is a sum cocone.

3a) A sketch *S* =(*G*, *D*, *L*, *K*) consists of graph *G*, a set *D* of diagrams in *G*, a set *L* of cones and a set *K* of cocones in *G*.
3b) A model *m* of a sketch *S* =(*G*, *D*, *L*, *K*) in a category *C* is a homomorphism from *G* to the underlying graph of *C* that takes every diagram in *D* to a commutative diagram, every cone in *L* to a limit cone and every cocone in *K* to a limit cocone.

Remark: Sketches have models although they are not theories!

Definitions (Guitart 1986):

1) An (abstract) sketch is a data ***S***=(S, P, Y) where S is a category, P is a family of distinguished cones on S and Y is a family of distinguished co-cones on S. A realisation R of ***S*** is continuous and co-continuous functor R:S-->SET (which sends every distinguished (co-)cone from P (Y) to a (co-)limit (co-)cone in SET.

2) A concrete sketch is a data ***S***=(S, P) where S is a graph and P is a family of distinguished cones on (S, SET). A realisation R of S is a functor R :S-->SET such that for all

$p = (p_i : V \dashrightarrow C_i)$ from P

Lim Hom(C_i, R) <--> Hom(V , R)

3) A category is *naturally* sketchable iff if it is equivalent to (S, SET) with S a sketch.

Theorem (Guitart&Lair) :

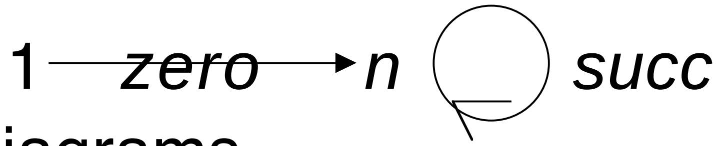
A category X is naturally sketchable iff it is equivalent to a category (S, SET) with S a *concrete* sketch.

Theorem (Barr&Wells):
Let *S* be a finite discrete sketch. Then there is a category *Th*(*S*) called <u>theory</u> of *S* and a model *u* : *S* -->*Th*(*S*) called <u>universal</u> such that for any any model *m* : *S* --> *C* there is a functor *f* : *Th*(*S*) --> *C* that preserves finite products and finite sums for which
(i) *fu = m*
(ii) if *f'* is another such functor then *f'* is naturally isomorphic to *f*.

Theorem (Barr):
Any finite *product* sketch has a model *i* called *initial* such that there is precisely one functor from *i* to every other model in the given category of models.

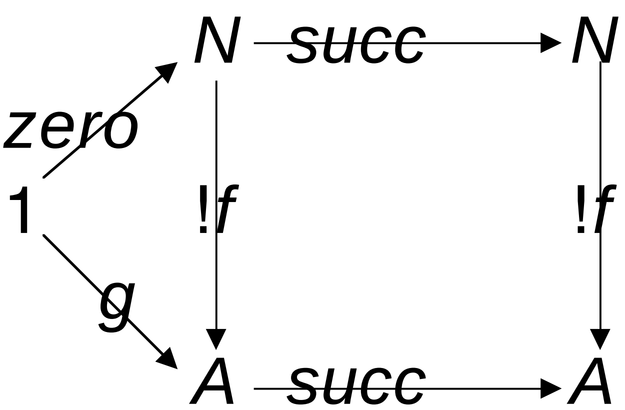
Ex.: a sketch for natural numbers

- nodes: 1, *n*
- cones with vertex 1 and empty base (this implies that in any model *m*(1) is terminal object)
- arrows: *zero*:1-->*n* , *succ*: *n*-->*n*



- no diagrams

Initial model in a (n abstract) category: natural number object, i.e. this diagram:



such that for any *A*, *g* there exist unique *f* making it commutative.

"Specifications in mathematics and computer science are most commonly expressed using a formal language with rules spelling out the semantics. However, there are other objects in mathematics intended as specifications that are not based on a formal language. Many of these are tuple-based; for example the signature of an algebraic structure or the tuple specifying a finite state automaton.
A sketch is another kind of formal abstract specification of a mathematical structure; it is based on a graph rather than on a formal language or tuple." (Barr&Wells 1990)

What is "formal and abstract" in a sketch?
Tentative answer: NOTHING

<u>Historical remark</u>: Before Hilbert Mathematics usually was not seen as *formal* science on a par with Logic. Logic in its turn did not reduce to *formal* logic. Confusion of mathematical and logical rigor with formal rigor occurred only in 20th century. These things must be properly distinguished.

Question: Are Euclidean "generic figures" (Circle and Straight Line), Lawvere’s “generic models” and sketches generic in a similar sense?

Tentative answer: YES

Some proposals:

- think of categorical constructions in terms of postulates rather than axioms.

Example: given morphisms *A*-->*B* and *B* -->*C* **to produce** morphism *A*-->*C*.

- Logic and truth-values may come about out of specific constructions rather then the other way round. In applications they can be empirically- and pragmatically-based.

- Abandon logical a priori after geometrical a priori.

Problem:
How strong the background category must be? Is the very notion of background indeed indispensable in any reasonable categorical setting?

Some conclusions:

- Reasonable foundations of mathematics involve constructive principles over and above "first truths". Logical truth-preserving inference plays a role in mathematical theories but can hardly be the *only* generic principle for such theories. The very idea of genericity appears to be more viable.
- Formal approaches shouldn't be identified with rigorous mathematical approaches. The simplistic scholastic metaphysics of form and content (form and matter) shouldn't be taken for granted in thinking about mathematical matters.
- Categorical logic and Sketch theory provide a new way of thinking about mathematical matters where the notions of *form* and *structure* don't play a major role. Structures are specific categories rather than the other way round.
- This new way of thinking shares with Euclidean thinking the idea of genericity.