# Univalent Foundations of Mathematics and Paraconsistency

Vladimir L. Vasyukov

**Abstract.** Vladimir Voevodsky in his Univalent Foundations Project writes that univalent foundations can be used both for constructive and for non-constructive mathematics. The last is of extreme interest since this project would be understood in a sense that this means an opportunity to extend univalent approach on non-classical mathematics. In general, Univalent Foundations Project allows the exploitation of the structures on homotopy types instead of structures on sets or structures on categories as in case of set-level mathematics or category-level mathematics. Non-classical mathematics should be respectively considered either as non-classical set-level mathematics or as non-classical category-level (toposes-level) mathematics. Since it is possible to directly formalize the world of homotopy types using in particular Martin-Lof type systems then the task is to pass to non-classical type systems e.g. da Costa paraconsistent type systems in order to formalize the world of non-classical homotopy types. Taking into account that the univalent model takes values in the homotopy category associated with a given set theory and to construct this model one usually first chooses a locally cartesian closed model category (in the sense of homotopy theory) then trying to extend this scheme for a case of non-classical set theories (e.g. paraconsistent ones) we need to evaluate respective non-classical homotopy types not in cartesian closed categories but in more suitable ones. In any case it seems that such Non-Classical Univalent Foundations Project should be directly developed according to Logical Pluralism paradigma and and it seems that it is difficult to find counter-argument of logical or mathematical character against such an opportunity except the globality and complexity of a such enterprise.

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Institute of Philosophy
Russian Academy of Science
Volkhonka 14
119991 Moscow
Russia
e-mail: vasyukov4@gmail.com

## 1. Introduction

A few years ago Vladimir Voevodsky have come up with an idea for a new semantics for dependent type theories - "univalent semantics" - which unlike of the usual semantics interpretation of types as sets interprets types as homotopy types. The key property of the univalent interpretation was that it satisfies the univalence axiom which makes it possible to automatically transport constructions and proofs between types which are connected by appropriately defined weak equivalences. According to Voevodsky [6] the key features of these "univalent foundations" are as follows:

1. Univalent foundations naturally include "axiomatization" of the categorical and higher categorical thinking.
2. Univalent foundations can be conveniently formalized using the class of languages called dependent type systems.
3. Univalent foundations are based on direct axiomatization of the "world" of homotopy types instead of the world of sets.
4. Univalent foundations can be used both for constructive and for non-constructive mathematics.

The central concept of the univalent foundations is a *homotopy*. A homotopy between continuous maps $f, g : X \to Y$ is a continuous map $\vartheta : X \times [0; 1] \to Y$ satisfying $\vartheta(x; 0) = f(x)$ and $\vartheta(x; 1) = g(x)$. Such a homotopy $\vartheta$ can be thought of as a "continuous deformation" of $f$ into $g$. Two spaces $X$ and $Y$ are said to be homotopy-equivalent if there are continuous maps going back and forth $X \underset{g}{\overset{f}{\rightleftarrows}} Y$, the compositions of which are homotopical to the respective identity mappings (which is tantamount to saying that there exist homotopies $fg \times [0; 1] \to 1_X$ and $gf \times [0; 1] \to 1_Y$). When this latter condition holds spaces $X$ and $Y$ are called *homotopy equivalent*, or interchangeably, *belonging to the same homotopy type*. It is natural to also consider homotopies between homotopies, referred to as *higher homotopies*. When we consider a space $X$, a distinguished point $p \in X$, and the paths in $X$ beginning and ending at $p$, and identify such paths up to homotopy, the result is the fundamental group $\pi(X; p)$ of the space at the point. If we remove the dependence on the base-point $p$ by considering the fundamental groupoid $\pi(X)$[1], consisting of all points and all paths up to homotopy. Next, rather than identifying

[1]A groupoid is like a group, but with a partially-defined composition operation Precisely, a groupoid can be defined as a category in which every arrow has an inverse. A group is thus a groupoid with only one object.

homotopic paths, we can consider the homotopies between paths as distinct, new objects of a higher dimension (just as the paths themselves are homotopies between points). Continuing in this way, we obtain a structure consisting of the points of $X$, the paths in $X$, the homotopies between paths, the higher homotopies between homotopies, and so on for even higher homotopies.

There is a groupoid model of Martin-Löf's type theory, where a given basic type $A$ (a judgements of the form $\vdash A : type$) is groupoid $A$, term $x$ of type $A$ ( judgement $\vdash x : A$) is object $x$ of groupoid $A$, and dependent type $B(x)$ (judgement $x : A \vdash B(x) : type$) is fibration[2] of the form $B \to A$. Identity type $Id_A(x; y)$ in this model is the *arrow groupoid* of groupoid $A$, which is a functor category of the form $[I; A]$ where $I$ is the connected groupoid having exactly two non-identical objects and a single non-identity isomorphism between these objects. The crucial idea here was to replace families of sets indexed by sets by families of groupoids indexed by groupoids.

Members of Voevodsky's hierarchy at low levels are as follows ($A$ is a space of $h$-level $n+1$ if for all its points $x; y$ path spaces $paths_A(x; y)$ are of $h$-level $n$):

- *Level 0*: up to homotopy equivalence there is just one contractible space[3] that we call "point" and denote *pt*;
- *Level 1*: up to homotopy equivalence there are two spaces at this level: the empty space $\varnothing$ and the point *pt*. We call $\varnothing; pt$ truth values; we also refer to types of this level as properties and propositions. Notice that $h$-level $n$ corresponds to the logical level $n-1$: the propositional logic (i.e., the propositional segment of our type theory) lives at $h$-level 1.
- *Level 2*: Types of this level are characterized by the following property: their path spaces are either empty or contractible. So such types are disjoint unions of contractible components (points), or in other words sets of points. This will be our working notion of set available in this framework.
- *Level 3*: Types of this level are characterized by the following property: their path spaces are sets (up to homotopy equivalence). These are obviously (ordinary at) groupoids (with path spaces hom-sets).
- *Level 4*: Here we get 2-groupoids.
- *Level* $n+2$: n-groupoids.

It is interesting to notice that like Euclid Voevodsky begins constructing his hierarchical universe of homotopy types with a point and then applies a simple inductive procedure for generating from this point the rest of this universe (cf.[3]).

The last feature of univalent foundations above looks very attractive for those who are interested in so-called non-classical mathematics paradigma which considers mathematics based on various non-classical logics. The last thesis becomes

[2]Fibrations here are functors $p : E \to B$ between grupoids $E, B$ such that for each object $e$ from $E$ and any isomorphism $i : p(e) \leftrightarrow b$ from $B$ there exists an isomorphism $j : e \leftrightarrow e'$ such that $p(j) = i$.

[3]A space $A$ is called *contractible* when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.

more comprehensive indeed if we take into account that such "mathematics" should obviously be non-constructive by their nature because within this paradigma they all differ drastically from mathematics based on intuitionistic logic.

At first glance it seems that considerations of such a kind are too abstract, too global and obscure to provide arguments for extending Voevodsky's program in such a manner. But they are indispensable for studies of the foundations of mathematics being the part of the quest for answering the question of proven uniqueness of mathematic existed which would not be taken for granted due to the lack of the worthwhile rivals. And here is one more tendency in univalent foundations program and like consisting not just in incorporating logic to mathematical structures but in deriiving logic from mathematical considerations, in an "internalization" of logic and making it the secondary thing. But why one believe that the result always will be the same? We will try to show that there are some other possibilities which deserve to be taken into account in future investigations.

## 2. Logical pluralism and non-classical mathematics

The motto of the first conference on non-classical mathematics (Hejnice, Czech republic, 2009) was: "The 20th century has witnessed several attempts to build (parts of) mathematics on different grounds than those provided by classical logic. The original intuitionist and constructivist renderings of set theory, arithmetic, analysis, etc. were later accompanied by those based on relevant, paraconsistent, non-contractive, modal, and other non-classical logical frameworks. The subject studying such theories can be called non-classical mathematics and formally understood as a study of (any part of) mathematics that is, or can in principle be, formalized in some logic other than classical".

The featured topics included in program of this conference, but were not limited to, were, in particular, the following:

- *Intuitionistic mathematics*: Heyting arithmetic, intuitionistic set theory, topos-theoretical foundations of mathematics, etc.
- *Constructive mathematics*: constructive set or type theories, pointless topology, etc.
- *Substructural mathematics*: relevant arithmetic, non-contractive naive set theories, axiomatic fuzzy set theories, etc.
- *Inconsistent mathematics*: calculi of infinitesimals, inconsistent set theories, etc.
- *Modal mathematics*: arithmetic or set theory with epistemic, alethic, or other modalities, modal comprehension principles, modal treatment of vague objects, modal structuralism, etc.

An issue arising in connection with these topics would be formulated as follows: it is evident that there are not one but many true mathematics (such point of view could be called a mathematical pluralism) but what is their mutual relationship - they are rivals, amicable with each other, complementary or mutually exclusive? It reminds us the situation with non-euclidean geometry when after Lobachevsky and Riemann discoveries it turns out to be tthat there are many equivalent systems of geometry and the matter is just their relatiionship. If to draw an analogy then taking into account modern state of affairs in field of geometry one can in case of non-classical mathematics suggests an opposition of classical and non-classical mathematics: is our mathematics globally classical and locally non-classical (that is, have nonclassical parts) or, vice versa, it is globally non-classical being at the same time locally classical?

Conception of mathematical pluralism is mostly inspired by the situation in logics. In modern philosophy of logic very popular is the point of view of correctness not one but a great number of true logics. Namely this standpoint is being known as logical pluralism. Contemporary debate has led to a re-examination of some older views, especially the pluralism resulting from Carnap's famous tolerance for different linguistic frameworks and, for example, the work of Scottish/French logician Hugh McColl (1837–1909), who some have claimed was an early logical pluralist.

But what is the impact of foundational logic (or rather the change of foundational logic) in non-classical mathematics on the mathematical constructions themselves? Does it really matter?

## 3. Paraconsistent sets and homotopies

On the one hand, we can say that foundational logic serves not so much "to prop up the house of mathematics as to clarify the principles and methods by which the house was built in the first place. 'Foundations' as a discipline that can be seen as a branch of mathematics standing apart from the rest of the subject in order to describe the world in which the working mathematician lives" [2, p. 14]. But, on the other hand, set theory being the *lingua universalis* for mathematical foundations grows on the base of foundational logic and as mathematical practice shows the change of logical basis is not unnoticed for set theory. As a consequence we have now intuitionistic set theory, paraconsistent set theory, fuzzy set theory, quantum set theory etc. nucleating the foundational frameworks of the respective non-classical mathematics.

The last thesis would be reinforced by the non-classical attitude in homotopy theory consideration. For this aim let us consider the definitions which take place in the usual homotopy type theory. But firstly we recall the inductive definition (see [7]) which is used for describing homotopy type hierarchy:

- $(i)$ Given space $A$ is called *contractible* when there is point $x : A$ connected by a path with each point $y : A$ in such a way that all these paths are homotopic.
- $(ii)$ We say that $A$ is a space of $h$-level $n + 1$ if for all its points $x; y$ path spaces $paths_A(x; y)$ are of $h$-level $n$.

This completes the defnition.

And now return to the main definitions [6, p.1]:

- A (homotopy) type $T$ is said to be of $h$-level 0 if it is contractible,
- A (homotopy) type $T$ is said to be of $h$-level 1 if for any two points of $T$ the space of paths between these two points is contractible,
- A (homotopy) type $T$ is said to be of $h$-level $n+1$ if for any two points of $T$ the space of paths between these two points is of $h$-level $n$.

Then we have:

- There is only one (up to a homotopy equivalence) type of $h$-level 0 - the one point type $pt$.
- There are exactly two types of $h$-level 1, $pt$ and $\emptyset$; i.e. types of $h$-level 1 are the truth values.
- Types of $h$-level 2 are types such that the space of paths between any two points is either empty or contractible. Such a type is a disjoint union of contractible components i.e. (up to an equivalence) types of $h$-level 2 are sets.
- Types of $h$-level 3 are (homotopy types of nerves of) groupoids.
- More generally, types of $h$-level $n+2$ can be seen as equivalence classes of $n$-groupoids.

For $\emptyset$ condition $(ii)$ is satisfied vacuously; for $pt$ $(ii)$ is satisfied because in $pt$ there exists only one path, which consists of this very point. Usually $\emptyset, pt$ are called *truth values* and also refer to types of this level as *properties* and *propositions*. Notice that $h$-level $n$ corresponds to the logical level $n-1$: the propositional logic (i.e., the propositional segment of our type theory) lives at $h$-level 1.

But what about $\emptyset$ ? It is known (see [1]) that there is a system $ZF_1$ of paraconsistent set theory that related to Church's version of Zermelo-Fraenkel set theory $ZF_0$ with a universal set as a da Costa paraconsistent first-order logic $C_1^=$ is related to the clasical first-order predicate calculus $C_0^=$. In essence, "$ZF_1$ should be 'partially' included in $ZF_0$, though the latter is is also to be contained, in a certain sense, in the former" [1, p. 170]. The basic set-theoretic concepts of $ZF_1$ are analogous to those of $ZF_0$, although the concepts involving negation give rise to two notions: one involving the weak negation $(\neg)$ and the other the strong negation $(\neg*)$. As a result we have, for instance, two empty sets: $\varnothing = \{x : x \neq x\}$ and $\varnothing^* = \{x : \neg * (x = x)\}$.

In this case (i.e. one will purport sets not from the class of $ZF_0$-models but from the class of $ZF_1$-models) the respective point from the definition above will be changed:

- There are exactly three types of $h$-level 1, $pt$,$\emptyset$ and $\varnothing^*$; i.e. types of $h$-level 1 are the *truth values*.

Of course, since it is known that each axiom scheme of $ZF_0$ generates two corresponding axiom schemes of $ZF_1$, one with the strong negation and another with the weak one, then we can say that $ZF_1$ simply includes $ZF_0$. Hence, it seems that our reformulation of the point above does not essentially distort the whole construction of usual homotopy theory. The only consequences is that the generality of this theory will be restricted.

But since $ZF_1$ should be considered as some extension of $ZF_0$ then one can assume that there are some (paraconsistent) sets which are outside the scope of usual set theory. Hence, our usual account of the category $Set$ as the category of *all* sets will be incomplete and $Set$ appears to be just a subcategory of some category $PSet$ which should include paraconsistent sets too.

Anyway, will such kind of limitation be capable to affect the project of univalent foundations? The answer depends not only on the level of logical and mathematical tools exploited but also on the alternatives proposed.

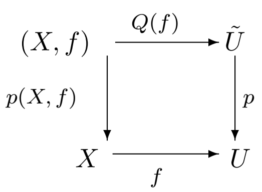
## 4. Paraconsistent categories and types

Along with set-theoretical there are another aspects of restriction of generality of homotopy type theory and respectively univalent foundations. Analyzing this program A.Rodin writes: "in Voevodsky's univalent foundations homotopy types turn to be the elementary bricks for constructing the whole of the mathematical universe including its logic" [3, p.224]. This inclusion having its effect in uniform hierarchical treatment of propositions simply as data of the specific type along with sets and categories. This uniformity plainly displayed in the semantic construction.

Syntactically a non-classical way of extending univalent foundations seems to be evident: one need to employ non-classical logical connectives and axioms in all formulations e.g. issue from typed $\lambda-$calculi with non-classical forming operations. But there are some deadends on this way - the open question of the general conception of negation forming operation (in many logical systems the negation is a primitive connective unlike the intuitionistic logic).

More challenging seems the semantic approach. To obtain a model with values in a category one need to construct a category with some additional structure, which is an object defned up to an equivalence. A technique for doing this which Voevodsky found very useful is based on the notion of a universe structure in a category [6, p. 4]. Let $C$ be a category. By a universe structure on $C$ we will mean a collection of data of the following form:

1. a final object $pt$ in $C$,
2. a morphism $p : \tilde{U} \to U$,
3. for any morphism $f : X \to U$ a choice of a pull-back square




Usually then it is postulated that $C$ is an *lccc* - a locally cartesian closed category. Doing this we obtain an opportunity to exploit a well known interpretation of logical connectives of intuitionistic logic as type forming operators since

Martin-Lof's intuitionistic type theory has an *lccc* as a category-theoretic model. What will happen if we modify $C$?

Early in [4] the intepretation of da Costa paraconsistent logics in topos of functors was proposed. As the basic construction there have been implemented so-called $CN$-categories. Generalizing their definition (original construction was based on pre-order categories) we can shortly characterize such a category $C$ as cartesian closed category for which the following conditions are fulfilled:

- for any object $a$ of $C$ there is an object $Na$ such that we have arrows $NNa \to a$ and $a^o \to (Na)^o$ in $C$ where $a^o = N\langle a, Na\rangle$ and for any arrow $d \to a$ there is an arrow $d \to Na$ in $C$;
- for any two objects $a, b$ in $C$ there is an arrow $a^o \to (b \Rightarrow a) \Rightarrow ((b \Rightarrow Na) \Rightarrow Nb)$ (here $\Rightarrow$ is an exponential);
- $1 \cong [a, Na]$ and $0 \cong \langle a^o, Na^o\rangle$.

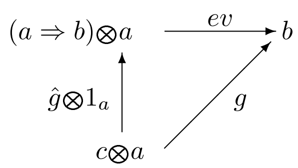
Let us call such category a paraconsistent cartesian closed category - *pccc*. The main advantage of *pccc* is that there is the categorical interpretation of paraconsistent negation in it (along with da Costa paraconsistent logics). What should be done as one more step in our consideration it is an introduction of *lpccc* - a locally paraconsistent cartesian closed category. And if we succeed then we can try to transform our category $C$ above in *lpccc*. Simultaneously one can ask the question: would it be right to speak in this case that we obtain in such a way a model of paraconsistent type theory? If yes then would we consider the situation from the point of view of non-classical univalent foundations project?

One more example would be obtained in a similar way by issuing from the categorical interpretation of relevant logic $R$ in topos of functors (see [5]). Here the original subject were so-called $RN$-categories which generalization in nutshell should be defined as $\otimes-$cartesian cliosed categories with negation (again by generalizing the original construction based on pre-order categories). In detail one can defined them as the bicomplete categories endowed with a covariant bifunctor $\otimes : C \times C \to C$ and a contravariant functor $N : C \to C$ such that the following conditions are satisfied:

**(i):** for any objects $a, b, c$ in $\mathcal{C}$ there are the following natural isomorphisms:

$a \otimes [b, c] \cong [a \otimes b, a \otimes c]$,
$[b, c] \otimes a \cong [b \otimes a, c \otimes a]$,

**(ii):** $C$ allows exponentiation relative to $\otimes$, i.e. the following diagram commutes




where $\Rightarrow$ is an exponential;

**(iii):** the following functorial equations are satisfied:

(a) $(g_1 f_1) \otimes (g_2 f_2) = (g_1 \otimes g_2)(f_1 \otimes f_2)$;
(b) $1_A \otimes 1_B = 1_{A \otimes B}$.

**(iv):** $C$ has an object 1 such that $1 \otimes a \cong a$ and there is an arrow $a \to a \cong 1$ in $C$ for all $a$ in $C$;
**(v):** for any objects $a, b, c$ in $C$, $a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$.
**(vi):** for any objects $a, b$ in $C$ there is an arrow $a \otimes b \to b \otimes a$.
**(vii):** for any object $a$ in $C$ there is an arrow $a \to a \otimes a$.
**(viii):** $N^2 a \cong a$ for any $a$ in $C$;
**(ix):** for any arrow $a \otimes b \to c$ there is an arrow $a \otimes Nc \to Nb$ in $C$.

If we denote such categories as relevant cartesian closed categories - *rccc* - and consider their more sophisticated version - a locally relevant cartesian closed categories *lrccc* - then one can conclude that we arrive at non-classical model of a hypothetical relevant type theory. Taking into account an internal paraconsistency of relevant logic $R$ which is now mirroring in the *lrccc* due to the existence of an interpretation of $R$ in topos of functors from $RN$-category to *Set* would it again be true to claim that there exists one more way to obtain a version of non-classical (paraconsistent) univalent foundations project?

## 5. Conclusion

The extremely non-classical point of view considered here seems to be too much marginal for taking him into account. But if the mathematical universe includes its logic in one or another way then it seems that logical pluralism inevitably will have direct influence on the mathematics. Hence, mathematical pluralism and emerging of non-classical mathematics are not accidental phenomena and these faces of plurality in mathematics are her real proper faces.

On the other hand, the implied lack of the uniqueness of univalent foundations project again make actual an issue of mathematical "paradise lost". It seems that we can turn out to be the witnesses of the process of paradigmal shifting (from set-theoretic to homotopy type paradigma). But is there one true mathematics still is an open question.

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Vladimir L. Vasyukov