Did Lobachevsky have a Model of his “Imaginary Geometry”?

Andrei Rodin (IPRAS and SPBU)

Kazan State University, December 2, 2021

Hilbert, *Foundations of Geometry* 1899 :

"Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters A, B, C,. . . ; those of the second, we will call straight lines and designate them by the letters a, b, c,.. ; and those of the third system, we will call planes and designate them by the Greek letters α, β, γ . [..] We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated", "between" ; "parallel", "congruent", "continuous", etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry. These axioms [..] express certain related fundamental facts of our intuition. "

Hilbert, Hilbert to Frege circa 1900 :

"You say that my concepts, e.g. "point", "between", are not unequivocally fixed [..]. But surely it is self-evident that every theory is merely a framework or schema of concepts together with their necessary relations to one another, and that basic elements can be construed as one pleases. If I think of my points as some system or other of things, e.g. the system of love, of law, or of chimney sweeps [..] and then conceive of all my axioms as relations between these things, then my theorems, e.g. the Pythagorean one, will hold of these things as well. In other words, each and every theory can always be applied to infinitely many systems of basic elements."
(*idem* for tables, chairs and beer mugs)

frame Axioms for Group Theory ("definition of group") :

G1 : $x \circ (y \circ z) = (x \circ y) \circ z$ (associativity of $\circ$)

G2 : there exists an item *1* (called *unit*) such that for all x
$x \circ 1 = 1 \circ x = x$

G3 : for all x there exists x^{-1} (called *inverse* of x) such that
$x \circ x^{-1} = x^{-1} \circ x = 1$.

Remark :

Group theory in the colloquial mathematicians' sense of the the term is not an axiomatic theory (in the usual logicians' sense) with axioms **G1**-**2** but a theory of models of such axioms. Each particular group (however identified) is a model of **G1**-**2**. But in the Group theory we study many groups, cf. Lagrange theorem.

If one restricts available models to set-theoretic ZF-based models of **G1**-**2** then one is in a position to claim that all theorems of Group theory are provable in ZF (as an axiomatic theory in logicians' sense).

Modern Axiomatic Method in Russia :

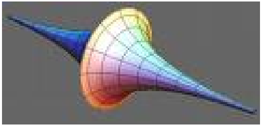
The pioneer : Veniamin Federovitch Kagan (1869-1953) : *Foundations of Geometry* in 2 volumes, Odessa, 1905-1907

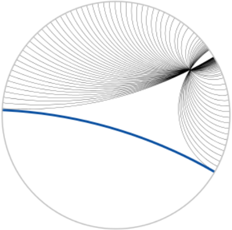
some disciples : Petr Konstantinovitch Rashevsky, Issaac Moiseevitch Yaglom, Viktor Vladimirovitch Wagner

Major achievement : Edition of Collected Works of Nikolai Ivanovitch Lobachevsky with Commentaries

Models of LG (anachronistically)

It is commonly known that Lobachevskian aka Hyperbolic 2D geometry has a number of *models* : Beltrami's Pseudosphere model (which is, in fact, only a *partial* model), Beltrami-Klein Projective model, Poincaré confomal disk and half-plane models, and some other.





Models of LG (anachronistically)

All such models are Euclidean in the following broad sense :

they interpret axioms of LG in an intuitive theoretical Euclidean background : either via a direct embedding into an Euclidean space (the case of pseudosphere) or indirectly via suitable geometric transformations (metric, projective and/or conformal as in the above examples).

Models of LG (anachronistically)

All such models are Euclidean in the following broad sense :

they interpret axioms of LG in an intuitive theoretical Euclidean background : either via a direct embedding into an Euclidean space (the case of pseudosphere) or indirectly via suitable geometric transformations (metric, projective and/or conformal as in the above examples).

An underlying epistemological reason behind this type of modelling seems to be Kantian : it is assumed that human intuitive spatial representation is Euclidean. So the best (and arguably the only) way to get an intuitive grasp on LG-plane is to represent it with some Euclidean constructions.

A model of EG-plane in LG-space ?

Are there Hyperbolic models of Euclidean geometry ? More concretely : is there a model of the Euclidean plane in the Hyperbolic 3-space ?

The question first appears as somewhat weird (particularly, against the aforementioned Kantian epistemological assumption), and the wanted model seems to be very exotic.

By my experience, it takes a while, even for an expert in the field, to figure out the correct positive answer — save the rare case when he or she is also an expert in the history of the subject.

a historical fact

As a matter of **historical fact**, the wanted "exotic" hyperbolic model of Euclidean plane was designed and used, independently, both by Nikolai I. Lobachevsky and János Bolyai. It played a key role in the discovery of Hyperbolic geometry (more below).

Hilbertian optics

The mismatch between the historical fact (anachronistically presented) and today's theoretical understanding is due to Hilbertian conceptual optics, which involves, in particular, the aforementioned Kantian assumption.

Hilbertian optics

The mismatch between the historical fact (anachronistically presented) and today's theoretical understanding is due to Hilbertian conceptual optics, which involves, in particular, the aforementioned Kantian assumption.

Without denying the theoretical significance of this conceptual optics, I claim that Lobachevsky's pre-Hilbertian way of reasoning, which led him to his discoveries, is important on its own right, and should not be seen merely as a preparatory step to Hilbert's axiomatic achievements.

Timeline (1792 - 1856) :

- 1823 : *Geometry* (Decimal angle measure : 100 degrees in d instead of 90 ; after Laplace ?)
- 1829-30 : *Foundations of Geometry*
- 1835 : *Imaginary Geometry*
- 1835-38 : *New Foundations of Geometry*
- 1836 : *Application of Imaginary Geometry to some Integrals*
- 1840 : *Studies in Theory of Parallels* (in German)
- 1855-56 : *Pangeometry* (French/Russian)

Major Influences : (after V.F. Kagan) :

Jean le Rond d'Alembert : La Encyclopédie de Diderot, 1757 (date of the first publishing of Geometry entry)

"As it shown in the AXIOM entry [of the Encyclopaedia] that these things are useless in all Sciences. It would be, therefore, very appropriate to eliminate axioms from the elements of Geometry disregarding the fact that these are still commonly used. [...]"

"The chimeric enterprise of looking for an *imaginary* rigour [in geometrical reasoning] is even worse than avoiding the *exact* rigour, which we promote here."
(Enc. Géométrie, my translation from French, my emphasis)

Major Influences : (after V.F. Kagan) :

An aspect of the "exactly rigourous" conceptual order in Geometry according to d'Alemebrt :

begin with solids then proceed to surfaces to lines to points via an iterated abstraction (with important reservations).

Remarkably, d'Alembert and his co-authors do not disapprove on the traditional concept of *elements* along with that of axiom ! The Encyclopaedia project to reform the standard Euclid-style Elements is highly non-trivial.

Major Influences : (after V.F. Kagan) :

Geometry textbooks inspired by the Encyclopaedia and available for Lobachevsky (also in Russian translations) :

- Legendre (since 1794)
- Bézout (since 1775)
- Lacroix (since 1795)

Main features of Lobachevsky's exposition of geometry (pro d'Alembert contra Euclid) :

Main features of Lobachevsky's exposition of geometry (pro d'Alembert contra Euclid) :

- "Geometry is a part of mathematics that treats measurements of space" (FG 1829)

Main features of Lobachevsky's exposition of geometry (pro d'Alembert contra Euclid) :

- "Geometry is a part of mathematics that treats measurements of space" (FG 1829)
- a reversal of Euclid's conceptual order : first solids (bodies), then surfaces and lines (via abstraction and the concept of *contact*).
  "Clearly, no mathematical science should begin with obscure concepts borrowed from Euclid" (FG 1829)

Main features of Lobachevsky's exposition of geometry (pro d'Alembert contra Euclid) :

- "Geometry is a part of mathematics that treats measurements of space" (FG 1829)
- a reversal of Euclid's conceptual order : first solids (bodies), then surfaces and lines (via abstraction and the concept of *contact*).
  "Clearly, no mathematical science should begin with obscure concepts borrowed from Euclid" (FG 1829)
- no explicit axioms ; the "exact rigour" of physical measurement and symbolic computation contra the "imaginary rigour" of Euclid-style axiomatic theory-building

Lobachevskian Parallels (after STP of 1840)

Lobachevsky makes a terminological change : he calls "parallels" (not just non-intersecting straight lines but) the two boundary lines which separate secants from non-secants (i.e. parallels in the usual terminology) passing through a given point.

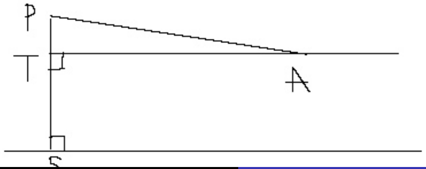
So in Lobachevsky's terms there exist exactly <u>two</u> parallels to a given straight lines passing through a given point, which may eventually coincide if AP holds (i.e. in the Euclidean case).

Lobachevskian Parallels (after STP of 1840)

$$\tan(\alpha/2) = e^{-d} \tag{1}$$

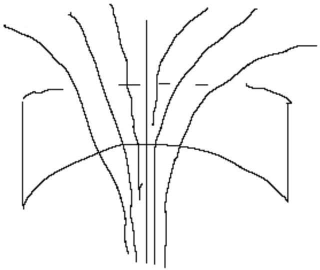
where

$$\alpha = \angle SPA;\ d = SP$$



Horosphere

Geodesics on a horosphere are (hyperbolic models of) Euclidean straight lines. The horosphere is a (full smooth) embedding of Euclidean plane into the hyperbolic 3-space.



Trigonometry : Euclidean, Spherical, Hyperbolic

spherical case :

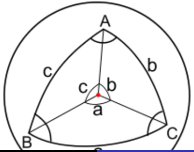
$$cos\ a = cos\ b \cdot\ cos\ c\ +\ sin\ b \cdot\ sin\ c \cdot\ cos\ A$$

$$cos\ A = -cos\ B \cdot\ cos\ C\ +\ sin\ B \cdot\ sin\ C \cdot\ cos\ a$$

hyperbolic case :

$$cos \rightarrow cosh; sin \rightarrow sinh; sin \cdot sin \rightarrow -sinh \cdot sinh$$

or equivalently $A \rightarrow iA$, etc., where $i = \sqrt{-1}$ (as rightly guessed by J. Lambert back in the 18th c.)



Remarks :

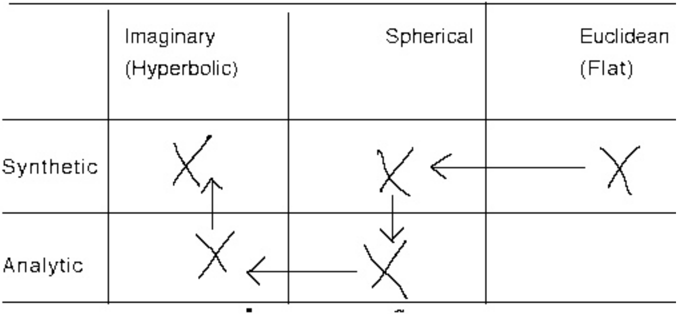
It is the development of the hyperbolic trigonometric calculus that allows us to see Lobachevsky (along with Bolyai) as a true discoverer of LG (see also Gray :1979). An independent application of this new calculus, but not the idea of producing geometrical worlds from a "pure" mathematical thinking, motivates the expression "imaginary geometry" in Lobachevsky.

Remarks :

It is the development of the hyperbolic trigonometric calculus that allows us to see Lobachevsky (along with Bolyai) as a true discoverer of LG (see also Gray :1979). An independent application of this new calculus, but not the idea of producing geometrical worlds from a "pure" mathematical thinking, motivates the expression "imaginary geometry" in Lobachevsky.

From a traditional logical point of view Lobachevsky's development of plane geometry via 3D geometry was a conceptual mistake. It was later corrected by H. Liebmann who in 1907 derived the same hyperbolic trigonometric identities by planar means. But in Lobachevsky's D'Alambertian view the right order of concepts is the opposite, so there is no mistake here.

"Relative consistency" argument



(1) What is a model ?

Today's standard Tarskian notion of *model* (of a given formal theory) combines two related things :

(1) What is a model ?

Today's standard Tarskian notion of *model* (of a given formal theory) combines two related things :

- translation (interpretation) of a given non-interpreted theory in(to) another (usually well-interpreted) theory ; example : Hilbert's arithmetical models of geometrical theories ;

(1) What is a model ?

Today's standard Tarskian notion of *model* (of a given formal theory) combines two related things :

- translation (interpretation) of a given non-interpreted theory in(to) another (usually well-interpreted) theory ; example : Hilbert's arithmetical models of geometrical theories ;
- intuitive support for (intuitive interpretation of) a given non-interpreted theory ; example : the "standard" model of PA ; the "traditional" model (?) of Euclidean geometry in Hilbert's axiomatic setting.

(1) What is a model ?

Lobachevsky's geometrical thinking suggests a sharper distinction between the two notions :

Lobachevsky's reasoning about/in the hyperbolic 3-space is intuitive and contentual from the outset ; it doesn't involve an interpretation of self-standing theoretical scheme in some pre-given spatial terms.

On the contrary, Lobachevsky does apply an interpretation of Euclidean plane in terms of hyperbolic curve surface, to wit the horosphere, in order to express hyperbolic functions in terms of familiar trigonometric and/or exponential functions.

(2) Geometrical Intuition and Physics

The popular Kantian assumption according to which human cognitive capacities are limited by the Euclidean mode of spatial representation is hardly justified.

Lobachevsky effectively extends the capacity of spatial representation beyond the Euclidean limit. Mathematical intuitions allow for a progressive development (via invention and further transmission and conservation in reproducible learning) along with mathematical theories. More precisely, such intuitions are proper elements of the corresponding theories and their models.

(2) Geometrical Intuition and Physics

New mathematical intuitions support new applications of mathematical theories in physics and other natural sciences.

Beltrami's identification of Lobachevsky's hyperbolic 3-space with a Rimannian 3-manifold of constant negative curvature in 1868 made full justice to the intuitive aspect of Lobachevsky's and Riemann's achievements. It helped Albert Einstein in 1915 to use the Riemannian geometry as a mathematical backbone of his General theory of Relativity.

Logical Positivism and Dialectical Materialism

The doctrine of Dialectical Materialism imposed in the USSR as a part of the official ideology of the state was more favourable to d'Alembert-style practically-oriented reform of mathematics and mathematical education than the contemporary popular doctrine of Logical Positivism, which in its turn favoured Hilbert-style (and later also the Bourbaki-style) logical foundations of mathematics.

Logical Positivism and Dialectical Materialism

This is in spite of the fact that d'Alembert's philosophy was qualified by Soviet Marxists as a variety of "mechanical", i.e., naive, non-reflected and underdeveloped variety of materialism.

Logical Positivism and Dialectical Materialism

This is in spite of the fact that d'Alembert's philosophy was qualified by Soviet Marxists as a variety of "mechanical", i.e., naive, non-reflected and underdeveloped variety of materialism.

The difference between the two philosophical doctrines reflects different understanding of modern mathematics including non-Euclidean geometry.

M. Stone 1961

arguing in favour of the Bourbaki-inspired reform of mathematical education in the US known as New Maths :

"While several important changes have taken place since 1900 in our conception of mathematics or in our points of view concerning it, the one which truly involves a revolution in ideas is the discovery that mathematics is entirely independent of the physical world."

M. Greenberg 1974

In his popular undergraduate geometry textbook (edition of 1974) Marvin Greenberg writes :

"This discovery [of non-Euclidean geometries] has had a liberating effect on mathematics, who now feel free to invent any set of axioms they wish and deduce conclusions from them. In fact, this freedom may account for the great increase in the scope and generality of modern mathematics."

Contrast : R. Feynman 1965

"When we come to consider the words and definitions which children ought to learn, we should be careful to to teach "just" words. [..]Many of the math books which are suggested now are full of such nonsense [..]. The words [..]should be the very same words used, *at least*, by the users of mathematics in the sciences, and in engineering."

Contrast : V. Arnold 1998

"Mathematics is a part of Physics. Physics is an experimental empirical science, a part of Natural Science. Mathematics is a part of Physics where experiments are cheap."

Even if Soviet mathematicians just like their Western colleagues expressed different views and attitudes to axiomatic foundations of mathematics in general and to Bourbaki's enterprise in particular, such an anachronistic (and historically definitely erroneous) description of Lobachevsky's discoveries could hardly be found in Soviet geometry textbooks and moreover in Soviet-time historical narratives, which by and large sticked to a "materialistic" interpretation of modern mathematics as an integral part of mathematically-laden natural science and technology.

From the dialectic-materialistic point of view, as I understand it, there is no dramatic difference between the Euclidean and Non-Euclidean geometrical theories as far as the role of these theories in physics and other mathematically-laden sciences is concerned.

In Soviet sources Lobachevsky's revolutionary discovery is stressed and dramatised in terms of Lobachevsky's decisive break with the "dogmatic" and more traditional Euclid-style axiomatic presentation of geometry but not in the anachronistic terms of Hilbertian "axiomatic freedom".

References I

- V.I. Arnold, *On mathematical teaching (in russian)*, Russian Math. Surveys, 53(1) :229–236, 1998.
- J. Gray, *Non-Euclidean Geometry : a re-interpretation*, Historia Matematica, vol. 6 (1979), p. 236-258.
- M.J. Greenberg, *Euclidean and Non-Euclidean Geometries*, Freeman & Company, W. H. (1974).
- R. Feynman, *New textbooks for the 'new' mathematics*, Engineering and Science, 28(6) :9–15, 1965.
- V.F. Kagan, *Foundations of geometry*, vol. 1 (Russian), Economics Publishing, Odessa, 1905.

References II

- V.F. Kagan. *Foundations of geometry*, vol. 2 (Russian), Economics Publishing, Odessa, 1907.
- V.F. Kagan, *Lobachevsky*, (in Russian), Academy of Sciences USSR Publishing, Moscow-Leningrad 1948.
- N.I. Lobachevski, *Geometrische Untersuchungen zur Theorie der Parallellinien*, Berlin : F. Fincke, 1840.
- A. Rodin, *Did Lobachevsky have a model of his Imaginary geometry ?*, Philosophy of Science (Novosibirsk), vol. 66, n. 3 (2015), p. 34-63, preprint arXiv :1008.2667
- M. Stone, *Revolution in Mathematics*, American Mathematical Monthly, 68(8) (1961), p. 715–734.

Thank You !

www.philomatica.org