19. ON THE INTERPRETATION OF INTUITIONISTIC LOGIC*

This paper can be considered from two completely different viewpoints.

1. If the intuitionistic cognitive presuppositions are not accepted, then one should take into account only the first section. The conclusions of this section can roughly be summarized as follows.

Along with the development of theoretical logic, which systematizes the schemes of proofs of theoretical truths, it is also possible to systematize the schemes of solution of problems, for example, geometrical construction problems. In this case the syllogism principle can be formulated, for example, as follows. *If we can reduce the solution of problem* b *to the solution of problem* a *and the solution of problem* c *to the solution of problem* b, *then the solution of* c *can also be reduced to the solution of* a.

By introducing an appropriate system of symbols, we can develop a formal calculus enabling us to construct symbolically systems of such solution schemes. Thus, a new *calculus of problems* arises along with theoretical logic. Note that in this case there is no need for special (for example, intuitionistic) cognitive presuppositions.

The following remarkable fact holds: *the calculus of problems coincides in form with the Brouwerian logic recently formalized by Heyting* [1], [2].

2. The second section in which the general intuitionistic presuppositions are accepted, presents a critical analysis of intuitionistic logic. It is shown that this logic should be replaced by the calculus of problems, since the objects under consideration are in fact problems, rather than theoretical propositions.

§1

We do not define the notion of a *problem* but explain it by means of some examples:

1. Find four integers x, y, z and n such that

$$x^n + y^n = z^n, \quad n > 2. \tag{1}$$

2. Prove that Fermat's theorem is false.

3. Construct a circle [1] passing through three given points (x, y, z).

* 'Zur Deutung der intuitionistischen Logik', *Math. Z.* **35** (1932), 58–65.

[1] To be precise, the permissible means of construction must be indicated when stating this problem.

4. Given one root of the equation $ax^2 + bx + c = 0$, find the other root.
5. Assuming that the number π has a rational expression

$$\pi = m/n$$

find a similar expression for the number e.

There is an obvious distinction between the first and the second problems which however, is not the subject of a specifically intuitionistic proposition. [2] The fourth and fifth problems are examples of *conditional* problems; the premise of the latter is false, and hence the fifth problem is meaningless or empty. Here and elsewhere the proof of the fact that a problem is *meaningless* is considered as a solution of this problem.

We believe that these examples and explanations allow us to use unambiguously the notions of "problem" and "solution of a problem" in all the cases encountered in specific fields of mathematics. [3] In what follows problems will be denoted by lower case italic letters $a, b, c, \ldots$.

If a and b are problems, then $a \wedge b$ denotes the problem "solve both problems a and b", while $a \vee b$ stands for "solve at least one of the problems a and b". Further, $a \supset b$ is the problem "given a solution to problem a, solve problem b" or, which is the same, "reduce the solution of problem b to the solution of problem a".

We never assume a problem to be solvable. Suppose, for example, that Fermat's theorem is true. Then the solution of the first problem is contradictory. Accordingly, $\neg a$ denotes the problem "assuming that there is a solution to problem a, derive a contradiction". [4]

According to these definitions, if a, b, c, d are problems, then each formula $p(a, b, c, \ldots)$ formed from the symbols $\wedge$, $\vee$, $\supset$, and $\neg$ also denotes a problem.

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[2] By contrast, from the point of view of classical logic the propositions "Fermat's theorem is not true" and "there exist four numbers satisfying (1)" are equivalent.

[3] The basic notions of the logic of propositions, "proposition" and "proof of a proposition" present the same difficulties.

[4] We note that $\neg a$ should not be read as the problem "prove the unsolvability of problem a". In the general case, if the "unsolvability of problem a" is considered as a completely defined notion, we only obtain that $\neg a$ implies the unsolvability of a, and not the converse assertion. If, for example, it were proved that a realization of the well-ordering of the continuum is beyond our possibilities, it would not be possible to assert that the existence of such a well-ordering implies a contradiction.

If $a, b, c, \ldots$ are merely symbols of undefined problems, then it can be said that $p(a, b, c, \ldots)$ is a function of the given variables $a, b, c, \ldots$. In the general case, if x is a variable (of any kind) and $a(x)$ denotes a problem whose meaning depends on the values of x, then $(x)a(x)$ denotes the problem "find a general method for solving the problem $a(x)$ for each specific value of x". This should be understood as follows: the problem $(x)a(x)$ is solved if the problem $a(x_0)$ can be solved for each given specific value of x_0 of the variable x by means of a finite number of steps which are fixed in advance (before x_0 is set). [5]

For a function $p(a, b, c, \ldots)$ of undefined problems a, b, c, $\ldots$ we simply write [6]

$$\vdash p(a, b, c, \ldots)$$

instead of

$$(a)(b)(c)\ldots p(a, b, c, \ldots).$$

Hence, $\vdash p(a, b, c, \ldots)$ denotes the problem "find a general method for solving the problem $p(a, b, c, \ldots)$ for each individual choice of the problems a, b, c, $\ldots$".

Problems of the form $\vdash p(a, b, c, \ldots)$ where p is expressed by means of the symbols $\vee, \wedge, \supset$, and $\neg$ constitute the subject of the *elementary calculus of problems.* [7]

The corresponding functions $p(a, b, c, \ldots)$ are *elementary problem functions.*

The fact that *I* have solved a problem is a purely subjective one of no general interest in itself. Logical and mathematical problems, however, possess a special property of *universal validity of their solutions*, that is, if *I* have solved a logical or a mathematical problem, then *I* can present this solution in a commonly accepted way, and this solution must *necessarily* be recognized as being correct, although this necessity is of a somewhat ideal nature since the reader is assumed to have adequate qualification. [8]

[5] As above, we hope that this definition will not lead to misunderstanding in specific areas of mathematics.

[6] This interpretation of the symbol $\vdash$ completely differs from that suggested by Heyting, although it leads to the same rules of the calculus.

[7] This definition is similar to that of the elementary propositional calculus. In propositional calculus, however, logical functions expressed by the symbols $\wedge$, $\vee$, $\supset$ and $\neg$ can be expressed in terms of two of the symbols. In the calculus of problems these four symbols are independent.

[8] The same is literally true for proofs of theoretical propositions. It is, however,

The aim of the calculus of problems is to develop a method allowing one to apply automatically a number of simple computational rules for solving a problem $\vdash p(a,b,c,\ldots)$ where $p(a,b,c,\ldots)$ is an elementary problem function. In order to reduce the whole problem to these computational rules we have, however, to assume that the solutions of certain elementary problems are already known. We *postulate* that the following two groups (A) and (B) have already been solved. The presentation below is meant only for a reader who has already solved all these problems. [9]

2.1 $\vdash a \supset a \wedge a;$

2.11 $\vdash a \wedge b \supset b \wedge a;$

2.12 $\vdash (a \supset b) \supset (a \wedge c \supset b \wedge c);$

2.13 $\vdash (a \supset b) \wedge (b \supset c) \supset (a \supset c);$

2.14 $\vdash b \supset (a \supset b);$

(A) 2.15 $\vdash a \wedge (a \supset b) \supset b;$

3.1 $\vdash a \supset a \vee b;$

3.11 $\vdash a \vee b \supset b \vee a;$

3.12 $\vdash (a \supset c) \wedge (b \supset c) \supset (a \vee b \supset c);$

4.1 $\vdash \neg a \supset (a \supset b);$

4.11 $\vdash (a \supset b) \wedge (a \supset \neg b) \supset \neg a.$

Thus, we assume that, given any problems a, b, c, the reader can solve all problems above behind the symbol $\vdash$. This presents no difficulties. For example, in problem 2.12, assuming that the solution of b has already been reduced to the solution of a, one should reduce the solution of $b \wedge c$ to that of $a \wedge c$. Let a solution of $a \wedge c$ be given. This means that we are given both a solution of a and a solution of c. By the hypothesis, we can derive a solution of b from that of a, and, since a solution of c is known, we obtain solutions of both problems b and c and hence a solution of problem $b \wedge c$.

essential that every proved proposition is *true*; for problems there is no such notion of truth.

[9] In the case of propositional calculus, if one wants to establish the truth of certain consequences of the axioms, one has to show first that the axioms are true. As to the numbering of the formulas, see [1].

This argument contains a general method for solving the problem

$$(a \supset b) \supset (a \wedge c \supset b \wedge c),$$

which is valid for any a, b, c. Thus, the problem

2.12 $\vdash (a \supset b) \supset (a \wedge c \supset b \wedge c)$

(under the generality symbol $\vdash$) can now be regarded as being solved.

The second group, (B), for which we have postulated the existence of solutions consists of only three problems. [10] Namely, our considerations are based on the fact that knowing a general method we can always solve the following problems for given elementary problem functions p, q, r, s, ...

I. If $\vdash p \wedge q$ is solved, solve $\vdash p$.

II. If $\vdash p$ and $\vdash p \supset q$ are solved, solve $\vdash q$.

III. If $\vdash p(a, b, c, \ldots)$ is solved, solve $\vdash p(q, r, s, \ldots)$.

We can now formulate the *rules* of our calculus of problems.

1. First, we include the problems of group (A) in the list of solved problems.

2. If the list includes $\vdash p \wedge q$, then we are allowed to replace it by $\vdash p$.

3. If both formulas $\vdash p$ and $\vdash p \supset q$ are in the list, then we can replace them by $\vdash q$.

4. If $\vdash p(a, b, c, \ldots)$ is in the list and $q, r, s, \ldots$ are arbitrary problem functions, then we are allowed to replace it by $\vdash p(q, r, s, \ldots)$ in the list.

Based on the above postulates, it is easily seen that the formal calculus does in fact guarantee the solution of the corresponding problems.

We are not going to develop this calculus further here, since all formal rules and *a priori* formulas above coincide with the computational rules and axioms suggested by Heyting [1]. Hence, we can interpret all formulas of this paper as problems and assume that all problems are solved.

Here we only note some particularly interesting problems (which are also regarded as being solved):

4.3 $\vdash a \supset \neg\neg a$;

4.2 $\vdash (a \supset b) \supset (\neg b \supset \neg a)$;

4.3.2 $\vdash \neg\neg\neg a \supset \neg a$.

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[10] These problems cannot, however, be expressed using symbols of the elementary calculus of problems.

The solutions of problems 4.3 and 4.2 are clear without calculation. The solution of problem 4.3.2 is obtained from 4.3 and 4.2 if b is replaced by $\neg\neg a$ in 4.2.

If the *a priori* accepted formulas in group (A) are supplemented by the formula

$$\vdash a \vee \neg a \qquad (2)$$

(which is the law of the excluded middle in propositional calculus), then we obtain the complete set of axioms of classical propositional calculus. In our interpretation of problems, formula (2) reads as follows: find a general method which for any problem a allows one either to find its solution or to derive a contradiction from the existence of such a solution! In particular, if the problem a consists of proving a proposition, one must have a general method which allows one either to prove each proposition or to reduce it to a contradiction. Unless the reader considers himself omniscient, he will perhaps agree that (2) cannot be in the list of problems that he has solved.

Astonishingly, however, the problem: [11]

4.8 $\vdash \neg\neg(a \vee \neg a)$

can be solved, as is seen from Heyting's calculus.

The formula

$$\vdash \neg\neg a \supset a$$

(known in propositional calculus as the law of double negation) cannot appear in our calculus of problems either, since in view of 4.8, it implies (2).

It is thus seen that in contrast to the formulas obtained in Heyting's intuitionistic logic, even some of the simplest formulas of classical propositional logic cannot appear in our calculus of problems.

It should also be mentioned that if $\vdash p$ is false in classical propositional logic, then the corresponding problem $\vdash p$ cannot be solved. Indeed, in view of the earlier accepted formulas and rules of the calculus of problems, this formula $\vdash p$ readily implies the contradictory formula $\vdash a \wedge \neg a$ (cf. [3]).

[11] In propositional logic, 4.8 represents Brouwer's theorem on the consistency of the law of excluded middle.

§2.

The intuitionistic criticism of logical and mathematical theories is based on the following principle: ***each meaningful proposition must indicate one or several completely definite situations accessible by our experience.*** [12]

If a is a general proposition of the type "each element of a set K possesses the property A" and if, moreover, K is an infinite set, then the negation of a, that is, the proposition "a is false", does not satisfy the stated principle. To overcome this difficulty, Brouwer suggests a new definition of negation, namely "a is false" should be understood as "a leads to a contradiction". Thus, the negation of a proposition a is transformed into the *existential sentence* "there exists a chain of logical inferences leading to a contradiction if a is assumed to be true".

Existential propositions were, however, profoundly criticised by Brouwer. Namely, from the intuitionistic point of view there is no sense in saying simply that "there is at least one element of a set K possessing a property A" without specifying that element.

Brouwer does not, however, intend to exclude existential propositions from mathematics completely. He only explains that an existential proposition should not be stated without presenting the corresponding construction. At the same time, according to Brouwer, an existential proposition is not a mere indication of the fact that we have already found the desired element of K. In this case the existential proposition would be false *prior to* the invention of the construction and true *after* that. Thus, propositions of a completely new type arise, which, although their content does not change in time, can nevertheless be stated only under certain conditions.

The natural question which can arise is whether this specific type of proposition is a mere fiction. Indeed, the problem "find an element of a set K possessing a property A" is posed. This problem actually has a certain sense independent of the state of our knowledge. If this problem has been *solved*, that is, if the corresponding element x is found, we obtain the empirical proposition "our problem is now solved". Thus, Brouwer's existential proposition is partitioned into two elements: an objective component (problem) and a subjective component (its solution). So there remains nothing that can be interpreted as

[12] Cf. [4]. The investigation below of negated and existential propositions is in essence close to this paper by Weyl.

an existential proposition in the proper sense of this term.

Therefore the major result of the intuitionistic criticism of negated propositions should be formulated in the following simple way: *in general it is meaningless to consider the negation of a general proposition as a definite proposition.* But then the subject of intuitionistic logic disappears, since the law of the excluded middle becomes true for all propositions whose negations make sense. [13]

As to mathematics, it follows that the solution of a problem must be considered as an independent task (along with the proof of theoretical propositions). As was shown in §1, the formulas of intuitionistic logic also acquire a new meaning in the area of problems and solutions. [14]

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References

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[13] There arises, however, a new question: which logical laws are true for propositions whose negations are meaningless.

[14] This interpretation of intuitionistic logic is closely related to Heyting's idea [5]. Heyting does not, however, distinguish explicitly between a proposition and a problem.