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Negated Problems According to Kolmogorov

In his paper “On the Interpretation of Intuitionistic Logic” first published in 1932 in German A.N. Kolmogorov proposes to interpret the propositional intuitionistic logic (eralier published by Heyting) as a Calculus of Problems. The concept of problem in Kolmogorov’s account is primitive and demonstrated with a number of examples. A special type of problems, which I shall call propositional problems) has the form “prove proposition P”. But there are also problems of different types like to “construct an equilateral triangle by ruler and compass”. In this context Kolmogorov distinguishes problem to “prove that Fermat statement is false” and from problem to “find a quadruple of natural numbers that satisfy Fermat condition” where Fermat statement says that Fermat condition is not satisfiable. The commonly known BHK semantics of intuitionsitic logic (so called after names of Brouwer, Heyting and Kolmogorov) does not support the above distinction.

How to interpret negation of problem P (in symbols \neg P)? One obvious suggestion is to interpret \neg P as problem to “prove that problem P is has no positive solution” where by positive solution one understands a fulfilment of request that constitutes the given problem such as constructing a geometrical figure, finding a quadruple of natural numbers satisfying certain condition, or proving a certain statement. But in Kolmogorov’s view this proposal is not tenable because it reduces general problems to propositional problems.

Recall that in the intuitionistic logic \neg P is introduced on the ground of derivation

P \vdash \bot

Under the usual intended interpretation of the above syntactic derivation P is a proposition and \bot is contradiction (falsum). Then a prove of \neg P amounts to assuming P and deriving a contradiction from this assumption. Kolmogorov grants that a proof of proposition “problem P has positive solution” proves \neg P (in words: P is unsolvable). But he insists that the converse does not hold: the fact that given problem P is unsolvable does not, generally, entail that the assumption that the wanted positive solution of P exists leads to a contradiction.

As an easy example think of regular heptagon, which is not constructible by ruler and compass while the assumption that such an object exists is not contradictory. This example can be challenged by arguing that the limitation of constructive means by ruler and compass is too restrictive and arbitrary. Then think of large cardinals that can be hardly constructed in any reasonable sense of the word but nevertheless can be thought of in an axiomatic setting without explicit contradiction.

A formal realisation of Kolmogorov's Calculus of Problems requires a framework that manages propositions and other types of mathematical objects simultaneously, and distinguishes between these different types of objects properly. A framework that satisfies such conditions is Homotopy Type theory (HoTT). I leave it as an open question whether there is a coherent interpretation of Kolmogorov's ideas about negated problems in HoTT.