Kolmogorov's Calculus of Problems and Homotopy Type theory.

Andrei Rodin

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Plan:

1. Problems and Theorems according to Kolmogorov

2. Constructive negation

3. Conclusions

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Zur Deutung der intuitionistischen Logik, *Mathematische Zeitschrift* 35 (1932)

Zur Deutung der intuitionistischen Logik.

Von

A. Kolmogoroff in Moskau.

Die vorliegende Abhandlung kann von zwei ganz verschiedenen Standpunkten aus betrachtet werden.

1. Wenn man die intuitionistischen erkenntnistheoretischen Voraussetzungen nicht anerkennt, so kommt nur der erste Paragraph in Betracht. Die Resultate dieses Paragraphen können etwa wie folgt zusammengefaßt werden:

Neben der theoretischen Logik, welche die Beweisschemata der theoretischen Wahrheiten systematisiert, kann man die Schemata der Lösungen von Aufgaben, z. B. von geometrischen Konstruktionsaufgaben, systematisieren. Dem Prinzip des Syllogismus entsprechend tritt hier z. B. das folgende Prinzip auf: *Wenn wir die Lösung von b auf die Lösung von a und die Lösung von c auf die Lösung von b zurückführen können, so können wir auch die Lösung von c auf die Lösung von a zurückführen.*

Man kann eine entsprechende Symbolik einführen und die formalen Rechenregeln für den symbolischen Aufbau des Systems von solchen Aufgabenlösungsschemata geben. So erhält man neben der theoretischen Logik eine neue *Aufgabenrechnung*. Dabei braucht man keine speziellen erkenntnistheoretischen, z. B. intuitionistischen Voraussetzungen.

Calculus of Problems 1932:

(1) "Along with the development of theoretical logic, which systematizes the schemes of proofs of theoretical results; it is also possible to systematize the schemes of solutions of problems, for example, geometric construction problems. [...] If we can reduce the solution of problem b to the solution of problem a, and the solution of problem c to the solution of problem c to the solution of problem b, then the solution of c can also be reduced to the solution of a.

Calculus of Problems 1932:

(2) "The following remarkable fact holds: the calculus of problems coincides in form with the Brouwerian logic recently formalized by Heyting" [reference to *Die formalen Regeln der intuitionistischen Logik*, 1930, in two parts]

Calculus of Problems 1932:

(3) "[Provided that] the general intuitionistic presuppositions are accepted . . . the intuitionistic logic . . . should be replaced by the calculus of problems, since the objects under consideration are in fact problems, rather than theoretical propositions. [the emphasis is mine]

Question: Is the difference between Kolmogorov's proposed interpretation of Heyting's calculus (= intuistionistic propositional calculus) in terms of *problems* and Heyting's own interpretation of this calculus in terms of *propositions* (Deutsch : *Aussagen*) essential or it is merely linguistic and superficial?

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In order to see how it works one needs to take into consideration the specific intuitionistic notion of proposition. I leave out here some historical details and explain here this intuitionistic notion after Per Martin-Löf (1984) and Dag Prawitz that will take us closer to the Homotopy Type theory.

The intuitionstic notion of proposition

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"When we hold a proposition to be true we make a *judgement*."

Alternative explanations / interpretations of judgements in MLTT

In MLTT(1984) there are four different forms of judgement; here is how Martin-Löf explains the judgement form $a\,A$ where A is a type and a is a term of this type; in fact, he proposes several apparently different readings of this formula and argues that they are essentially the same :

1. a is an element of set A
2. a is a proof (witness, evidence) of proposition A
3. a is a method of fulfilling (realising) the intention (expectation) A
4. a is a method of solving the problem (doing the task) A

Propositions and Sets

'If we take seriously the idea that a proposition is defined by lying down how its canonical proofs are formed [. . .] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set [. . .] apart. Instead we simply identify them, that is, treat them as one and the same notion."

Kolmogorov to Heyting, October 12, 1931

Every proposition p in your conception is, in my view, one of these two sorts:

α) p expresses the expectation that in such and such circumstances a [mathematical] experiment will give a determined result (for exemple, that an attempt to decompose an even number n into a sum of two prime numbers p, q will be successful if all pairs (p, q), where $p < n$ and $q < n$, are used. Every such "experiment" should be, of course, realisable with a finite number of well-defined operations.

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I prefer to reserve the name of proposition (Aussage) only to propositions of type α) and call "propositions" of type β) simply problems. Given a proposition p one has problems $\neg p$ (to reduce p to contradiction) and $+p$ (to prove p).

Examples of problems (Kolmogorov 1932):

1. Find integers x, y, z, n such that (R) $x^n + y^n = z^n$ and $n > 2$; [type β)]
2. Prove that LTF is false. [forme $\neg p$ where p is of type α, i.e. a proposition]

Kolmogorov's comment of 1985:

"On the interpretation of intuitionistic logic" was written with the hope that the logic of solutions of problems would later become a regular part of courses on logic. It was intended to construct a unified logical apparatus dealing with objects of two types - propositions and problems.

Kolmogorov to Heyting, continued.

[F]or case β) the difference between p and $+p$ is not essential, but the proposition $\neg\neg p \to p$ should not be seen as evident. In case α), on the contrary, p and $+p$ have different meaning, but one has $\vdash \neg\neg p \to p$ and $\vdash \neg\neg p \to +p$.

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Cf. Sergey Melikhov, A Galois connection between classical and intuitionistic logics, parts 1-2, arXiv:1312.2575, arXiv:1504.03379

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- all higher identity proofs are interpreted as higher homotopies;

Homotopical hierarchy of types for judgement $a\ A$

Definition: S is a space of h-level $n+1$ if for all its points x, y path spaces $x \ =_S \ y$ are of h-level n. where h-level is read as as the homotopy level.

- h-level (-2): single point pt;
- h-level (-1): the empty space $\emptyset$ and the point pt: truth-values aka (mere) propositions
- h-level 0: sets (discrete point spaces)
- h-level 1: flat path groupoids : no non-contractibe surfaces
- h-level 2: 2-groupoids : paths and surfaces but no non-contractible volumes
-
- h-level n: n-groupoids
- ...
- h-level ω: ω-groupoids

A top-down cumulative character of the homotopical hierarchy

Every k-type is a n-type for all $n > k$.

Every proposition is a set (either the empty set or a singleton), every set is a trivial flat groupoid (without paths save reflections), every flat groupoid is a trivial 2-groupoid (without path homotopies), etc.

Truncation (m<k)

$T^k \to T^m, m < k$

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The homotopical hierarchy of types is at odds with Martin-Löf's view according to which propositions, sets and higher-order constructions are essentially the same. It supports Kolmogorov's view on problems and propositions provided one counts higher-order homotopy types as constructions. I submit that such an interpretation of homotopy types in the context of constructive theories is coherent.

Negated Problems according to Kolmogorov 1932

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Question: Contradiction is, *prima facie*, a propositional notion. If contradiction is understood as usual then the above definition is in odds with Kolmogorov's strategy to keeping propositions and problems apart.

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Negated Problems, continued

The following remark of Kolmogorov hardly wholly clarifies the issue:

"We note that $\neg A$ should not be read as "prove the unsolvability of problem A". In the general case, if the "unsolvability of problem A" is considered as a completely defined notion, we only obtain that $\neg A$ implies the unsolvability of A but not the converse assertion. If, for example, it were proved that a realization of the well-ordering of the continuum is beyond our possibilities, it would not be possible to assert that the existence of such a well-ordering implies a contradiction. "

Negated Problems, continued

Corollary: If proposition "problem A has a (positive) solution" implies contradiction, then A is unsolbable. But the converse does not hold.
Another example (?): trisect a given angle by ruler and compass. (Should the "our possibilities" be understood in an absolute sense or it is a matter of convention like in the case of ruler and compass?)

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Remark: If problem A has a form "prove proposition P" than the unsolvability of A (i.e. the unprovability of P) does not imply that P is contradictory (and hence false): this is Gödel's syntactic incompleteness. How to interpret this feature for problems of type β?

Kolmogorov's critique of the standard intuitionistic negation

Brouwer suggests a new definition of negation, namely "*A* is false" should be understood as "*A* leads to a contradiction". Thus, the negation of a proposition *A* is transformed into an *existential sentence* "there exists a chain of logical inferences leading to a contradiction if *a* is assumed to be true. Existential sentences were, however, profoundly criticized by Brouwer. "

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Remark (Kolmogorov): If one takes into consideration only problems A such that $A \to \bot$ is constructively provable, then the Law of Excluded Middle holds, and gets back the Classical propositional logic.

HoTT with constructive negation?

HoTT in its existing forms involves the *single* empty type $\bot$ that is propositional, i.e., of *h*-level (-1). Because of the cumulative character of the *h*-hierarchy $\bot$ also qualifies as a higher type of any order.

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Thus HoTT in its existing form(s) does not allow for a representation of Kolmogorov's notion of unsolvable problem via a more constructive and more specific conception of constructive negation.

Constructive negation in an elementary setting: Heyting 1925

The relation of *apartness* for points:

1. $A \neq B \rightarrow B \neq A$
2. $\neg(A \neq B) \rightarrow A = B$
3. $A \neq B \rightarrow (C \neq A \vee C \neq B)$

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Cf. Michael Shulman, *Affine Logic for Constructive Mathematics* (July 2020), arXiv:1805.07518v2

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In order to realise Kolmogorov's notion of unsolvable problem / constructive negation in HoTT its underlying type theory needs to possess a hierarchy of "empty types", one for each *h*-level (on the top of "the" usual empty type at the bottom).

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It may be useful in applications since the real life the (un)solvability of a problems is typically a resource-related issue, and rarely an "absolute" notion related to logical consistency. To develop a type theory with such a feature is an interesting open problem.

Conclusion 1

Kolmogorov's idea to distinguish between problems and propositions (theorems), and build a unified logical framework for both without dispensing with their differences, if considered against the intuitionist strategy to merge these things into one generic notion of proposition, receives an unexpected support from the homotopical interpretation of Martin-Löf's constructive type theory originally developed according to this strategy.

Conclusion 2

Kolmogorov's treatment of unsolvable Problems contains logical ideas that has been not fully realised yet in a formal setting, and thus can serve as a motivation of new research, in particular, in HoTT and related areas.

THANKS!