How Mathematical Concepts Get Their Bodies: The Example of Forcing''

A precise outlook at the Kantian distinction between mathematical concepts and corresponding intuitions ("intuitus vel conceptus") reveals some details, which Kant himself mostly left unnoticed. Along with classical cases where the two basic aspects of mathematical thinking perfectly fit together there are also multiple examples when they don't. The first group of such examples comprises clear mathematical intutions, which remain only poorly conceptualised; the second group comprises well-formed mathematical concepts, which are poorly intuited. Abusing Kant's original meaning of the term one can describe certain examples of the second kind as ideal elements of mathematical thinking.

The historical development of mathematics shows a permanent trade going on between concepts and intuitions. It involves both the progressive conceptualisation of earlier acquired intuitions and the progressive intuiting of earlier acquired concepts. The latter process can be called desidealisation because it results into the situation when ideal elements of mathematical thinking acquire stronger intuitions and so loose their ideal character. Metaphorically speaking, desidealisation allows mathematical concepts to "get their bodies".

The aim of this paper is to justify the above broad picture through a number of historical examples. More specifically I shall describe the example of forcing, which seems me important not only for understanding the recent past of mathematics but also for conceiving of its possible near future.

A progressive conceptualisation of earlier developed intuitions can be observed in the history of geometry from its early days to mid-19th century. Greek geometry as we know it after Euclid, strictly speaking, allows only for objects constructible by ruler and compass. At the same time the Euclidean setting immediately makes one think of geometrical objects, which (as we know today) cannot be produced in this way. Although such objects in many cases can be provided with precise definitions, their conceptualisation in the given setting remain essentially incomplete. Descartes in his "Geometry" greatly enlarged the domain of well-conceptualised geometrical objects by suggesting a very different way of their conceptualisation. The Cartesian geometrical domain includes, in particular, all algebraic curves. But once again it leaves out a large class of objects which are immediately intuited but only poorly conceptualised in the given context. Any non-algebraic curve is an example. It was not earlier than in 19th century when the progress in geometrical conceptualisation reversed the situation and brought about certain geometrical concepts like that of projective space or Lobachevskian space which didn't fit earlier developed intuitions.

An obvious example of desidealisation concerns the very notion of ideal in its technical algebraic sense. The modern mathematical notion of ideal stems via Dedekind from Kummer's notion of ideal number. Kummer introduced this latter notion in a purely predicative way by studying rings of cyclotomic integers: he merely stipulated objects having certain desired properties without trying to build them out of available materials and without questioning their existence. Dedekind in his turn identified Kummer's ideal numbers with certain infinite classes (sets) of cyclotomic integers. Today's notion of algebraic ideal mimics Dedekind's but refers to an abstract ring rather than a ring based on complex numbers. As far as new generations of mathematicians habituated themselves to thinking about infinite sets as intuitively conceivable entities (by the analogy with finite sets) the notion of algebraic ideal compleately lost its ideal character.

The latter example has in fact a general significance. For Set theory can be seen as an universal semantic tool available to produce a model (i.e. a "body") for any sound mathematical concept on likes. Morover according to an often-repeated argument the existing mathematics doesn't use in this way but a very small amount of available sets. So one might suggest that in a Set-theoretic framework ideal elements are no longer possible. Nevertheless at least one such example is found in the Set theory itself, namely the construction made by Paul Cohen in 1963 with a new method he called forcing. Cohen's task was to produce a model of ZFC violating Continuum Hypothesis (CH), which says that the power-set PN of set N of all finite ordinals has the cardinality equal to the second infinite cardinal ALEF1. The idea of the desired construction is the following. In some model M of ZF one takes N, cardinal ALEF2 bigger than ALEF1, and considers some function F: N x ALEF2--> {0, 1}. This F can be seen as a set of functions N-->{0,1}, which has cardinality ALEF2; since every function N-->{0,1} detrmines a particular subset of N the existence of F guarantees that (in M) PN is strictly bigger than ALEF1, and so CH is violated. However it turns out that the desired F cannot be fully described by "constructive" axioms of ZF like the union or powerset axiom. For "constructible universe" L comprising only sets, which can be fully described in this explicit way, is a model of ZF where CH holds. Cohen's forcing amounts to adjunction to a given model M a new generic set A (not from a larger model which would confuse cardinalities but) as a merely formal symbol; then one writes down (by means of ZF) a set of formulae called forcing conditions, which express desired properties of A (albeit don't determine it completely). As Cohen himself repeatedly pointed out this construction is analogous to that of extension of a given field by new elements absent from this initial field but nevertheless described in its terms (think again of cyclotomic numbers): in both cases formal expressions with new "ideal" elements are made into elements of a new extended structure. Notice that the sense of "ideal" in Cohen's case is in fact stronger than in Kummer's case since no single formula of ZF describes A completely. For a suggestive analogy think about real numbers in terms of their rational approximations. The method of forcing allowed Cohen to get the desired model of ZFC violating CH.

Category theory, which nowadays progressively replaces the membership-based Set theory in its role of universal mathematical language, allows for a desidealisation of Cohen's construction by providing it with a clear geometrical sense. Forcing conditions form a small category F (in Cohen's case this is a partial order, i.e. a category having no more than one morphism for any pair of objects). The category (Fop, S) of all contravariant functors (pre-sheaves) from F to the category of sets S is a topos, i.e. it has properties similar (but not identical) to those of S. This implies that in (Fop, S) forcing conditions "get completed" and play the role of truth-values; one can interpret them in this setting as "stages of knowledge" about the hypothetical set A. The next step amounts to singling out a Boolean topos B of sheaves from the obtained topos of presheaves. This time one thinks of elements of F as open sets; the desired sheaves are presheaves, which respect the given topology. The obtained Boolean topos is ("almost") a model of ZF, which comprise sets determined through forcing conditions. The topology of this topos provides an intuitive sense for the whole construction and allows one to explain its features, which in the original exposition look miraculous.

My concluding remark is the following. Set-theoretic mathematics devoloped a radically new type of mathematical intuition related to the notion of infinite set and discqualified more traditional types of mathematical intuition related to our thinking of space and time (as described in particular by Kant). Among some other negative consequences this abrupt change of the earlier established practice significantly weakened links between foundations of mathematics and foundations of physics. The above example of forcing shows that category-theoretic foundations of mathematics may allow for providing today's and future mathematics with a stronger intuitive support related to our thinking of space, time, change and motion. I certainly don't mean here to revive Kantian philosophy of mathematics in full but its part concerning the role of intuition in mathematical thinking seems me still viable.