Euclid's Elements Forever

Andrei Rodin

philomatica.org

April 1, 2024

Plan:

1. Historical Background
2. The Text and Its Transmission
3. Features
   - Problem 1.1
   - Theorem 1.5

The Author

The Author

- **When:** $\alpha\kappa\mu\eta$ between - 347 (Plato's death) and -287 (Archimedes' birth) during the reign of Ptolemy I (367-283 B.C.).

The Author

- **When:** $\alpha\kappa\mu\eta$ between - 347 (Plato's death) and -287 (Archimedes' birth) during the reign of Ptolemy I (367-283 B.C.).
- **Where:** Alexandria (Egypt, founded in -331 by Alexander the Great)

The Author

- **When:** $\alpha\kappa\mu\eta$ between - 347 (Plato's death) and -287 (Archimedes' birth) during the reign of Ptolemy I (367-283 B.C.).
- **Where:** Alexandria (Egypt, founded in -331 by Alexander the Great)
- some anecdotes and bon mots like "there is no royal road in geometry" (says Euclid to Ptolemy according to Proclus; the same story is told by Stobaeus about Menaechmus and Alexander)

Works

Works

- extant: *Elements*, *Data*, *On Divisions* ($\pm$), *Optics*, *Phaenomena* (spherical astronomy)

Works

- extant: *Elements*, *Data*, *On Divisions* ($\pm$), *Optics*, *Phaenomena* (spherical astronomy)
- lost: *Conics*, *Pseudaria* (on fallacies in mathematical reaosning), *Porisms* (intermediate between problems and theorems), *Surface loci* (theory of surfaces?)

The Elements

The Elements

- $\sigma\tau o\iota\chi\epsilon\iota\alpha$ - the alphabet, the ABC (of mathematics)

The Elements

- $\sigma\tau o\iota\chi\epsilon\iota\alpha$ - the alphabet, the ABC (of mathematics)
- Euclid's Elements are not the first Elements ever composed; Proclus (5th c.) mentions Elements by Leon composed in Plato's circle (-4th c.)

John Murdoch (1927 - 2010):

Any attempt to plot the course of Euclid's Elements from the third century B.C. through the subsequent history of mathematics and science is an extraordinary difficult task. No other work — scientific, philosophical, or literary — has, in making its way from antiquity to the present, fallen under an editor's pen with anything like an equal frequency. And with good reason: it served, for almost 2000 years, as the standard text of the core of basic mathematics. As such, the editorial attention it constantly received was to be expected as a matter of course.

Transmission : a warning

Copies and re-editions of the Elements very rarely aimed at a literal transmission of Euclid's text or its accurate translation into a different language (with some exceptions pointed to below).

More often than not mathematicians attempted to improve on the original text of the Elements rather than preserve and transmit it in its authentic form. That makes a significant difference with how philosophical, literary or religious texts have been transmitted.

The text : a warning

As a result until the beginning of the 20th century "Euclid's Elements" did not refer to Euclid's original text as we know it today according to the best available historical knowledge. This is why saying that Euclid's Elements served as a standard mathematical texbook for many centuries is not accurate; it is a historical confusion in a sense. These textbooks changed its content (and at a smaller degree also their form) continuously throughout the history just as they do this today.

I'll say first few words about what is the original Euclid's Elements according to the best of our today's knowledge, and then come back to the transmission.

Urtext

Urtext

- Euclidis Opera omnia. Ediderunt I. L. Heiberg et H. Menge. (Lipsiae, B. G. Teubneri, 1883-1916). Sketch of Heiberg's philological proof:

Urtext

- Euclidis Opera omnia. Ediderunt I. L. Heiberg et H. Menge. (Lipsiae, B. G. Teubneri, 1883-1916). Sketch of Heiberg's philological proof:
- Theon of Alexandria (c. 335-405 A.D.): –> Simon Grynaeus (Basel 1533): *editio princeps*; Theon: "the last proposition of Book 6 is mine"

Urtext

- Euclidis Opera omnia. Ediderunt I. L. Heiberg et H. Menge. (Lipsiae, B. G. Teubneri, 1883-1916). Sketch of Heiberg's philological proof:
- Theon of Alexandria (c. 335-405 A.D.): –> Simon Grynaeus (Basel 1533): *editio princeps*; Theon: "the last proposition of Book 6 is mine"
- Francois Peyrard in 1808 discovered in Vatican library a MS without that proposition. Heiberg argued that this MS derives from a pre-Theonian source.

Urtext

- Euclidis Opera omnia. Ediderunt I. L. Heiberg et H. Menge. (Lipsiae, B. G. Teubneri, 1883-1916). Sketch of Heiberg's philological proof:
- Theon of Alexandria (c. 335-405 A.D.): –> Simon Grynaeus (Basel 1533): *editio princeps*; Theon: "the last proposition of Book 6 is mine"
- Francois Peyrard in 1808 discovered in Vatican library a MS without that proposition. Heiberg argued that this MS derives from a pre-Theonian source.
- M. Klamroth argues that at least some of Arabic translations of the Elements are based on more reliable sources. This discussion continues but the current consensus is on Heiberg's side.

Transmission before 15th c.

Transmission before 15th c.

- Greek-Romanian phase: Boethius (480-524 A.D.): Latin translation of the Elements (extant only in short fragments

Transmission before 15th c.

- Greek-Romanian phase: Boethius (480-524 A.D.): Latin translation of the Elements (extant only in short fragments
- First translation into Arabic circa 800 A.D. and 300 years of further work. Murdoch: "The Arabic phase of the Elements' history may well prove to be not merely the most manifold but, even mathematically, the most creative of all."
- Arabic-Latin phase: 12-15th centuries.

Transmission 15-16th c.

Transmission 15-16th c.

- Campanus of Novara (Venice 1482)

Transmission 15-16th c.

- Campanus of Novara (Venice 1482)
- Bartalameo Zamberti (Venice 1505)

Transmission 15-16th c.

- Campanus of Novara (Venice 1482)
- Bartalameo Zamberti (Venice 1505)
- the influence of the *editio princeps* Grynaeus (Basel 1533) is weak

Transmission 15-16th c.

- Campanus of Novara (Venice 1482)
- Bartalameo Zamberti (Venice 1505)
- the influence of the *editio princeps* Grynaeus (Basel 1533) is weak
- Commandino: Urbino 1575 Latin based on Greek

Transmission 15-16th c.

- Campanus of Novara (Venice 1482)
- Bartalameo Zamberti (Venice 1505)
- the influence of the *editio princeps* Grynaeus (Basel 1533) is weak
- Commandino: Urbino 1575 Latin based on Greek
- Clavius: Rome 1574: an edition rather than translation: 486 propositions by "Euclid" plus 671 wholly new propositions)

Transmission 17-18 c : some examples

Transmission 17-18 c : some examples

- Andreé Tacquet: Antwerp 1654: Elementa geometriae planae et solidae (original treatise). Became "Euclid" in the second edition in 1725.

Transmission 17-18 c : some examples

- Andreé Tacquet: Antwerp 1654: Elementa geometriae planae et solidae (original treatise). Became "Euclid" in the second edition in 1725.
- Isaac Barrow: Cambridge 1655: Euclidis Elementorum libri XV breviter demonstrati: pursuing the historical authenticity via a radical mathematical modernisation

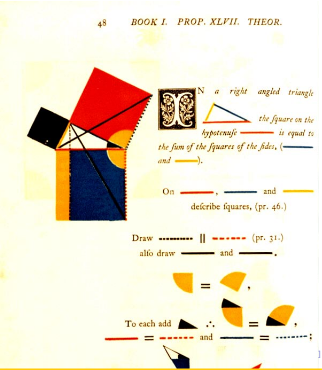
Transmission 17-18 c : some examples

- Andreé Tacquet: Antwerp 1654: Elementa geometriae planae et solidae (original treatise). Became "Euclid" in the second edition in 1725.
- Isaac Barrow: Cambridge 1655: Euclidis Elementorum libri XV breviter demonstrati: pursuing the historical authenticity via a radical mathematical modernisation
- Girolamo Saccheri: Milan 1733: Euclides ab omni naevo vindicatus (!): an attempt to prove the 5th Postulate, eventually led the non-Euclidan geometries.

Transmission 19th c

Murdoch: “a veritable avalanche of Euclid primers, frequently radically divergent from any imaginable text of the Elements”

Oliver Byrne, The First Six Books of the Elements of Euclid in which coloured diagrams and symbols are used instead of letters for the greater ease of learners, 1847



Translations from Heiberg's text

Translations from Heiberg's text

- Thomas L. Heath, The thirteen books of Euclid's Elements (Cambridge: Cambridge University Press, 1908)

Translations from Heiberg's text

- Thomas L. Heath, The thirteen books of Euclid's Elements (Cambridge: Cambridge University Press, 1908)
- Начала Евклида. / Пер.с греч. и комм. Д. Д. Мордухая-Болтовского под ред. М. Я. Выгодского и И. Н. Веселовского. М.-Л.: ГИТТЛ 1948-1950

Separation of Mathematics from Its History

Heiberg's historical edition of the Elements marks a definite (?) split between:

Separation of Mathematics from Its History

Heiberg's historical edition of the Elements marks a definite (?) split between:

- A historical study of Euclid's Elements and its multiple transformations and modifications (the latter remains understudied).

Separation of Mathematics from Its History

Heiberg's historical edition of the Elements marks a definite (?) split between:

- A historical study of Euclid's Elements and its multiple transformations and modifications (the latter remains understudied).
- and a further modernisation of the Elements: (1) D. Hilbert, Foundations of Geometry 1899, (2) Bourbaki, Elements of Mathematics 1939 -

Separation of Mathematics from Its History

Heiberg's historical edition of the Elements marks a definite (?) split between:

- A historical study of Euclid's Elements and its multiple transformations and modifications (the latter remains understudied).
- and a further modernisation of the Elements: (1) D. Hilbert, Foundations of Geometry 1899, (2) Bourbaki, Elements of Mathematics 1939 -
- Since Euclid's Elements are not the first and unique Elements ever written, studying its transformation is at least as important as studying of Euclid's original text or arguably even more important.

Separation of Mathematics from Its History

Heiberg's historical edition of the Elements marks a definite (?) split between:

- A historical study of Euclid's Elements and its multiple transformations and modifications (the latter remains understudied).
- and a further modernisation of the Elements: (1) D. Hilbert, Foundations of Geometry 1899, (2) Bourbaki, Elements of Mathematics 1939 -
- Since Euclid's Elements are not the first and unique Elements ever written, studying its transformation is at least as important as studying of Euclid's original text or arguably even more important.
- A new computer-based version of Bourbaki for the 21st century? (Voevodsky)

D. Hilbert, Grundlagen 1899:

GRUNDLAGEN DER GEOMETRIE

VON

DR. DAVID HILBERT,

O. PROFESSOR AN DER UNIVERSITÄT GÖTTINGEN.

ZWEITE, DURCH ZUSÄTZE VERMEHRTE UND MIT FÜNF ANHÄNGEN VERSEHENE AUFLAGE.

MIT ZAHLREICHEN IN DEN TEXT GEDRUCKTEN FIGUREN.

D. Hilbert, Grundlagen 1899:

So fängt denn alle menschliche Erkenntnis mit Anschauungen an, geht von da zu Begriffen und endigt mit Ideen.

Kant, Kritik der reinen Vernunft, Elementarlehre 2. T. 2. Abt.

Einleitung.

Die Geometrie bedarf — ebenso wie die Arithmetik — zu ihrem folgerichtigen Aufbau nur weniger und einfacher Grundsätze. Diese Grundsätze heißen **Axiome** der Geometrie. Die Aufstellung der Axiome der Geometrie und die Erforschung ihres Zusammenhanges ist eine Aufgabe, die seit *Euklid* in zahlreichen vortrefflichen Abhandlungen der mathematischen Literatur[1]) sich erörtert findet. Die bezeichnete Aufgabe läuft auf die logische Analyse unserer räumlichen Anschauung hinaus.

Die vorliegende Untersuchung ist ein neuer Versuch, für die Geometrie ein **vollständiges** und **möglichst einfaches** System von Axiomen aufzustellen und aus denselben die wichtigsten geometrischen Sätze in der Weise abzuleiten, daß dabei die Bedeutung der verschiedenen Axiomgruppen und die Tragweite der aus den einzelnen Axiomen zu ziehenden Folgerungen möglichst klar zu Tage tritt.

D. Hilbert, Grundlagen 1899:

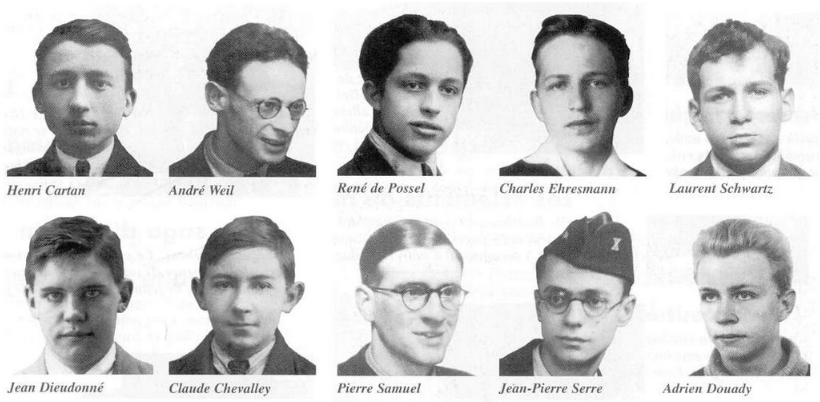
§ 1.

Die Elemente der Geometrie und die fünf Axiomgruppen.

Erklärung. Wir denken drei verschiedene Systeme von Dingen: die Dinge des ersten Systems nennen wir *Punkte* und bezeichnen sie mit $A, B, C, \ldots$; die Dinge des zweiten Systems nennen wir *Gerade* und bezeichnen sie mit $a, b, c, \ldots$; die Dinge des dritten Systems nennen wir *Ebenen* und bezeichnen sie mit $\alpha, \beta, \gamma, \ldots$; die Punkte heißen auch die *Elemente der linearen Geometrie*, die Punkte und Geraden heißen die *Elemente der ebenen Geometrie* und die Punkte, Geraden und Ebenen heißen die *Elemente der räumlichen Geometrie* oder *des Raumes*.

Wir denken die Punkte, Geraden, Ebenen in gewissen gegenseitigen Beziehungen und bezeichnen diese Beziehungen durch Worte wie „liegen", „zwischen", „parallel", „kongruent", „stetig"; die genaue und vollständige Beschreibung dieser Beziehungen erfolgt durch die *Axiome der Geometrie*.

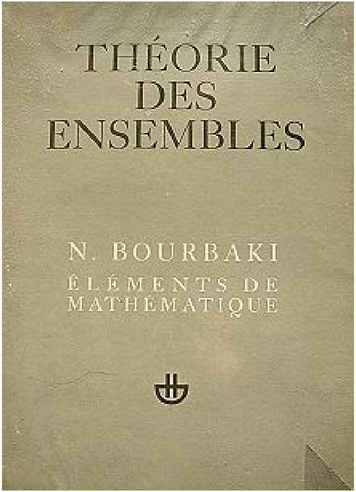
Bourbaki:



Henri Cartan, *André Weil*, *René de Possel*, *Charles Ehresmann*, *Laurent Schwartz*

Jean Dieudonné, *Claude Chevalley*, *Pierre Samuel*, *Jean-Pierre Serre*, *Adrien Douady*

Elements 1939 -



Structure

Structure

- Books 1-4: Basic geometry of Ruler of Compass

Structure

- Books 1-4: Basic geometry of Ruler of Compass
- Books 5-6: Geometric theory of proportion

Structure

- Books 1-4: Basic geometry of Ruler of Compass
- Books 5-6: Geometric theory of proportion
- Books 7-9: Arithmetic (including the arithmetical theory of proportion)

Structure

- Books 1-4: Basic geometry of Ruler of Compass
- Books 5-6: Geometric theory of proportion
- Books 7-9: Arithmetic (including the arithmetical theory of proportion)
- Book 10: Irrationalities

Structure

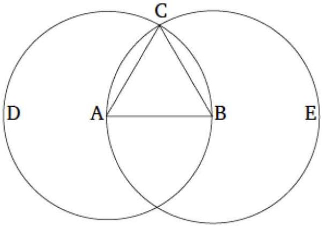
- Books 1-4: Basic geometry of Ruler of Compass
- Books 5-6: Geometric theory of proportion
- Books 7-9: Arithmetic (including the arithmetical theory of proportion)
- Book 10: Irrationalities
- Books 11-13: Stereometry

P1.1: enunciation:

To construct an equilateral triangle on a given finite straight-line.

P1.1: exposition:

Let AB be the given finite straight-line.



P1.1: specification:

So it is required to construct an equilateral triangle on the straight-line AB.

P1.1: construction:

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B [Post. 1].

P1.1: proof:

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [Axiom 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

P1.1: conclusion:

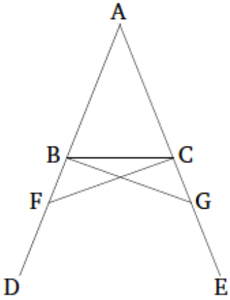
Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

T1.5: enunciation:

For isosceles triangles, the angles at the base are equal to one another, and if the equal straight lines are produced then the angles under the base will be equal to one another.

T1.5: exposition:

Let ABC be an isosceles triangle having the side AB equal to the side AC; and let the straight lines BD and CE have been produced further in a straight line with AB and AC (respectively). [Post. 2].



T1.5: specification:

I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.

T1.5: construction:

For let a point F be taken somewhere on BD, and let AG have been cut off from the greater AE, equal to the lesser AF [Prop. 1.3]. Also, let the straight lines FC, GB have been joined. [Post. 1]

T1.5: proof 1:

In fact, since AF is equal to AG, and AB to AC, the two (straight lines) FA, AC are equal to the two (straight lines) GA, AB, respectively. They also encompass a common angle FAG. Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG, and AFC to AGB.

T1.5: proof 2:

And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG [Ax.3]. But FC was also shown (to be) equal to GB. So the two (straight lines) BF, FC are equal to the two (straight lines) CG, GB respectively, and the angle BFC (is) equal to the angle CGB, while the base BC is common to them. Thus the triangle BFC will be equal to the triangle CGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus FBC is equal to GCB, and BCF to CBG.

T1.5: proof 3:

Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF, within which CBG is equal to BCF, the remainder ABC is thus equal to the remainder ACB [Ax. 3]. And they are at the base of triangle ABC. And FBC was also shown (to be) equal to GCB. And they are under the base.

T1.5: conclusion:

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

THANKS!